

OPTIMAL CONTROL AND HIGHER-ORDER MECHANICS FOR SYSTEMS WITH SYMMETRIES

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ABSTRACT. In this paper we will develop and design numerical methods for optimal control problems related to a class of underactuated Lagrangian mechanical systems where the configuration manifold is a trivial principal bundle. We will construct these geometric integrators using discrete variational calculus, that is deriving a discrete version of the higher-order Euler-Lagrange equations on trivial principal bundles. The analysis applies to systems subject to higher-order constraints (that is, depending on higher-order derivatives as, for example, the acceleration). Interesting applications as, for instance, a discrete derivation of the Euler-Lagrange equations for higher-order Lagrangians and higher-order reduced Lagrangians, respectively, are shown. We find interesting applications both in the optimal control of an underactuated vehicle and the well-known plate ball problem, the latter seen as an optimization problem with nonholonomic constraints.

1. INTRODUCTION

In the last years, the study of Lagrangian reduction and reconstruction for mechanical systems with Lie group symmetries has drawn an extraordinary interest in different areas of science as engineering, economy, physics, etc. The goal of this paper is to study, from a geometric point of view, variational integrators for optimal control problems of mechanical systems defined on a trivial principal bundle and its applications in optimal control theory. Our motivation is the control of the class of underactuated mechanical systems. This type of mechanical systems, i.e. the underactuated, is characterized by the fact that there are more degrees of freedom than actuators.

The presence of underactuated mechanical systems is ubiquitous in engineering applications as a result, for instance, of design choices motivated by the search of less cost devices or as a result of a failure regime in fully actuated mechanical systems. The underactuated systems include spacecraft, underwater vehicles, mobile robots, helicopters, wheeled vehicles, mobile robots, underactuated manipulators, etc.

We extend the theory of discrete mechanics based on discrete variational calculus for systems which depend on higher-order derivatives and subject to constraints (also depending of higher-order derivatives). We use Hamilton's principle and Lagrange multiplier theorem on manifolds in order to obtain discrete paths that approximately satisfy the dynamics and the constraints. This is achieved by formulating a higher-order discrete variational problem subject to higher-order constraints. Such formulation gives us the preservation of important geometric properties of the mechanical system, such as momentum, symplecticity, group structure, good behavior of the energy, etc [23].

A typical optimal control problem on a trivial principal bundle consists on finding a trajectory of the state variables and controls $(q(t), g(t), u(t))$ given fixed initial and final conditions $(q(0), g(0))$ and $(q(T), g(T))$ respectively, and, as well, minimizing the cost functional defined by

$$J(u, T) = \int_0^T \|u(t)\|^2 dt;$$

here, $q(t)$ belongs to the configuration space Q , $g(t)$ evolves on a Lie group G and $u(t)$ on the space of admissible controls, usually a subspace of \mathbb{R}^m . In the previous definition of the cost functional we are considering the final time T as a variable in the optimization problem. Nevertheless, we shall usually consider T fixed.

Our approach is based on recently developed structure-preserving numeric integrators for optimal control problems (see [17, 18, 33, 34, 35, 45] and references therein) based on solving a discrete optimal control problem as a discrete higher-order variational problem with higher-order constraints (see [4, 18, 17] for the continuous case) which are used for simulating and controlling the dynamics for satellites, spacecrafts, underwater vehicles, mobile robots, helicopters, wheeled vehicles, etc [9].

Finally, the case when the system is subject to nonholonomic constraints is also of great interest in optimal control of mechanical systems. Systems which nonholonomic constraints are carefully studied in [4, 5, 9, 29]. Some geometric integrators for this type of systems are developed in [13, 20, 21, 25, 43]. Thus, we also consider how our framework on discrete variational integrators with constraints is a suitable framework for optimal control of discrete nonholonomic mechanical systems. The class of systems that we study in this paper includes wheeled vehicles, such as robots on wheels and/or tracks. As an example, we pay attention to the optimal control of an homogeneous ball rotating in a plate.

1.1. Organization of the paper. The paper is structured as follows. In §2 we recall some results about variational integrators, Hamilton's principle on Lie groups and the discrete Euler-Poincaré equations. In §(3) we derive the continuous second-order Euler-Lagrange equations for trivial principal bundles from Hamilton's principle. Moreover, we study the higher-order case. The proposed method appears in §4 where we construct from a discretization of the Lagrangian and through discrete variational calculus the discrete higher-order Euler-Lagrange equations on trivial principal bundles. These equations are derived using a discrete Hamilton's principle. In the last section, we study continuous and discrete higher-order mechanics for systems subject to higher-order constraints. We apply these techniques in §(5), to the optimal control of underactuated mechanical systems, e.g. the optimal control of systems defined on the Lie group $SE(2)$, and finally, we study the optimal control of the plate ball problem.

2. DISCRETE MECHANICS AND VARIATIONAL INTEGRATORS

Let Q be a n -dimensional differentiable manifold, the configuration manifold, with local coordinates (q^i) , $1 \leq i \leq n$. Denote by TQ its tangent bundle with

induced coordinates (q^i, \dot{q}^i) . Given a Lagrangian function $L : TQ \rightarrow \mathbb{R}$, the *Euler-Lagrange equations* are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0, \quad 1 \leq i \leq n. \quad (1)$$

These equations are a system of n implicit second order differential equations.

In the sequel, we will assume that the Lagrangian is *regular*, that is, the matrix $\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right)$ is non-singular. From Lagrange's theorem, it is well known that the origin of these equations is variational (see [1, 41]).

The variational integrators [42] are derived from a discrete variational principle. These integrators also retain some of main geometric properties of the continuous system, such as symplecticity, momentum conservation and a good behavior of the energy associated with the Lagrangian system (see [23] and references therein). In the sequel we will review the construction of this type of geometric integrators.

A *discrete Lagrangian* is a map $L_d : Q \times Q \rightarrow \mathbb{R}$, which may be considered as an approximation of the integral action defined by a continuous Lagrangian $L : TQ \rightarrow \mathbb{R}$,

$$L_d(q_0, q_1) \approx \int_0^h L(q(t), \dot{q}(t)) dt,$$

where $q(t)$ is a solution of the Euler-Lagrange equations for L ; $q(0) = q_0$, $q(h) = q_1$ and the time step $h > 0$ is small enough.

Define the *action sum* $\mathcal{A}_d : Q^{N+1} \rightarrow \mathbb{R}$, corresponding to the Lagrangian L_d by

$$\mathcal{A}_d = \sum_{k=1}^N L_d(q_{k-1}, q_k),$$

where $q_k \in Q$ for $0 \leq k \leq N$, where N is the number of steps. The discrete variational principle then requires that $\delta \mathcal{A}_d = 0$ where the variations are taken with respect to each point q_k , $1 \leq k \leq N-1$ along the path, and the resulting equations of motion; (a system of difference equations) given fixed endpoints q_0 and q_N , are

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0, \quad (2)$$

where D_1 and D_2 denote the derivative to the Lagrangian respect the first and second arguments, respectively.

These equations are usually called *discrete Euler-Lagrange equations*. Under some regularity hypotheses (the matrix $(D_{12} L_d(q_k, q_{k+1}))$ is regular), it is possible to define a (local) discrete flow $\Upsilon_{L_d} : Q \times Q \rightarrow Q \times Q$, by $\Upsilon_{L_d}(q_{k-1}, q_k) = (q_k, q_{k+1})$ from (2).

We introduce now the two discrete Legendre transformations associated to L_d :

$$\begin{aligned} \mathbb{F}^- L_d : Q \times Q &\rightarrow T^*Q \\ (q_0, q_1) &\mapsto (q_0, -D_1 L_d(q_0, q_1)), \end{aligned} \quad (3)$$

$$\begin{aligned} \mathbb{F}^+ L_d : Q \times Q &\rightarrow T^*Q \\ (q_0, q_1) &\mapsto (q_1, D_2 L_d(q_0, q_1)), \end{aligned}$$

and the discrete Poincaré-Cartan 2-form $\omega_d = (\mathbb{F}^+ L_d)^* \omega_Q = (\mathbb{F}^- L_d)^* \omega_Q$, where ω_Q is the canonical symplectic form on T^*Q . ω_d is a symplectic form if the discrete Lagrangian is regular, which is indeed equivalent to $\mathbb{F}^- L_d$ (or $\mathbb{F}^+ L_d$) being a local diffeomorphism [42].

The discrete algorithm determined by Υ_{L_d} preserves the (pre-)symplectic form on $T^*(Q \times Q)$, ω_d , i.e., $\Upsilon_{L_d}^* \omega_d = \omega_d$. Moreover, if the discrete Lagrangian is invariant under the diagonal action of a Lie group G , then the discrete momentum map $J_d: Q \times Q \rightarrow \mathfrak{g}^*$ defined by

$$\langle J_d(q_k, q_{k+1}), \xi \rangle = \langle D_2 L_d(q_k, q_{k+1}), \xi_Q(q_{k+1}) \rangle$$

is preserved by the discrete flow. Therefore, these integrators are symplectic-momentum preserving. Here, ξ_Q denotes the fundamental vector field determined by $\xi \in \mathfrak{g}$, where \mathfrak{g} is the Lie algebra of G ,

$$\xi_Q(q) = \left. \frac{d}{dt} \right|_{t=0} (\exp(t\xi) \cdot q)$$

for $q \in Q$ (see [42] for more details).

2.1. Discrete Mechanics on Lie groups: Euler-Poincaré equations. In this section we recall the basics on discrete mechanics on Lie groups and Hamilton's principle on Lie groups for the formulation of the Euler-Poincaré equations.

If the configuration space is a Lie group G , then the discrete trajectory is represented numerically using a set of $N + 1$ points (g_0, g_1, \dots, g_N) with $g_i \in G$, $0 \leq i \leq N$.

A way to discretize a continuous problem is using a *retraction map* $\tau: \mathfrak{g} \rightarrow G$ which is an analytic local diffeomorphism which maps a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of the neutral element $e \in G$. As a consequence, it is possible to deduce that $\tau(\xi)\tau(-\xi) = e$ for all $\xi \in \mathfrak{g}$. The retraction map is used to express small discrete changes in the group configuration through unique Lie algebra elements (see [34]), namely $\xi_k = \tau^{-1}(g_k^{-1}g_{k+1})/h$, where $\xi_k \in \mathfrak{g}$. That is, if ξ_k were regarded as an average velocity between g_k and g_{k+1} , then τ is an approximation to the integral flow of the dynamics. The difference $g_k^{-1}g_{k+1} \in G$, which is an element of a nonlinear space, can now be represented by the vector ξ_k , in order to enable unconstrained optimization in the linear space \mathfrak{g} for optimal control purposes.

It will be useful in the sequel, mainly in the derivation of the discrete equations of motion, to define the *right trivialized* tangent retraction map as a function $d\tau: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ by

$$T\tau(\xi) \cdot \delta = Tr_{\tau(\xi)} d\tau_\xi(\delta),$$

where $\delta \in \mathfrak{g}$ and $r: G \times G \rightarrow G$ is the usual right translation on the group. Here we use the following notation, $d\tau_\xi := d\tau(\xi): \mathfrak{g} \rightarrow \mathfrak{g}$. The function $d\tau$ is linear in its second argument. From this definition the following identities hold (see [7] for further details):

Proposition 2.1. Given a map $\tau: \mathfrak{g} \rightarrow G$, its right trivialized tangent $d\tau_\xi: \mathfrak{g} \rightarrow \mathfrak{g}$ and its inverse $d\tau_\xi^{-1}: \mathfrak{g} \rightarrow \mathfrak{g}$, are such that for $g = \tau(\xi) \in G$ and $\eta \in \mathfrak{g}$, the following holds

$$\partial_\xi \tau(\xi) \eta = d\tau_\xi \eta \tau(\xi) \quad \text{and} \quad \partial_\xi \tau^{-1}(g) \eta = d\tau_\xi^{-1}(\eta \tau(-\xi)).$$

The most natural example of retraction map is the exponential map at the identity e of the group G , $\exp_e : \mathfrak{g} \rightarrow G$. We recall that, for a finite-dimensional Lie group, \exp_e is locally a diffeomorphism and gives rise a natural chart [40]. Then, there exists a neighborhood U of $e \in G$ such that $\exp_e^{-1} : U \rightarrow \exp_e^{-1}(U)$ is a local \mathcal{C}^∞ diffeomorphism. A chart at $g \in G$ is given by $\Psi_g = \exp_e^{-1} \circ \ell_{g^{-1}}$, where ℓ denotes the usual left-translation of an element of the group $\ell : G \times G \rightarrow G$.

In general, it is not easy to work with the exponential. For instance, if we are considering matrix groups, the right trivialized derivative and its inverse are defined by infinite series

$$d\exp_x y = \sum_{j=0}^{\infty} \frac{1}{(j+1)!} \text{ad}_x^j y, \quad d\exp_x^{-1} y = \sum_{j=0}^{\infty} \frac{B_j}{j!} \text{ad}_x^j y,$$

where B_j are the Bernoulli numbers, $x, y \in \mathfrak{g}$ and $\text{ad}_x y = [x, y]$ is the usual matrix bracket (see [23]). Typically, these expressions are truncated in order to achieve a desired order of accuracy.

In consequence it will be useful to use a different retraction map. More concretely, the Cayley map (see [7, 23] and the Appendix for further details) will provide us a proper framework in the examples shown below.

The following theorem, regardless of the *retraction structure* locally relating G and \mathfrak{g} , gives us the relation between the discrete Euler-Lagrange equations and the discrete Euler-Poincaré equations.

Theorem 2.1. [39] *Let G be a Lie group and $L_d : G \times G \rightarrow \mathbb{R}$ be a discrete Lagrangian function. We suppose that L_d is left-invariant over the diagonal action; that is, $L_d(gg_k, gg_{k+1}) = L_d(g_k, g_{k+1})$ with $g \in G$. Let $\widehat{L}_d : G \rightarrow \mathbb{R}$ be the restriction to the identity (that is, $\widehat{L}_d : (G \times G)/G \simeq G \rightarrow \mathbb{R}$, $\widehat{L}_d(g_k^{-1}g_{k+1}) = L_d(g_k, g_{k+1})$). For a pair of points $(g_k, g_{k+1}) \in G \times G$, we consider $W_k = g_k^{-1}g_{k+1}$. Then the following assertions are equivalent:*

1) $(g_k)_{k=0}^N$ extremize the discrete action $\sum_{k=0}^{N-1} L_d(g_k, g_{k+1})$ for all variation with initial and final fixed points.

2) The discrete Euler-Poincaré equations $r_{W_k}^* \widehat{L}'_d(W_k) - \ell_{W_{k-1}}^* \widehat{L}'_d(W_{k-1}) = 0$, $k = 1, \dots, N$ hold, where ℓ and r are the left- and right-translation of the Lie group and $'$ denote the partial derivative.

3) $(W_k)_{k=0}^N$ extremize $\sum_{k=0}^{N-1} \widehat{L}_d(W_k)$ for all variations $\delta W_k = -\eta_k W_k + W_k \eta_{k+1}$ with $\eta_0 = \eta_N = 0$; where $\eta_k \in \mathfrak{g}$ is given by $\eta_k = g_k^{-1} \delta g_k$.

The previous assertions are also equivalent to say that $(g_k)_{k=0}^N$ satisfy the Euler-Lagrange equations for L_d .

3. HIGHER-ORDER EULER-LAGRANGE EQUATIONS ON TRIVIAL PRINCIPAL BUNDLES

3.1. Higher-order tangent bundles. In this subsection we recall some basic facts of the higher-order tangent bundle theory. At some point, we will particularize this construction to the case when the configuration space is a Lie group G . For more details see [10, 36].

Let Q be a differentiable manifold of dimension n . It is possible to introduce an equivalence relation in the set $C^k(\mathbb{R}, Q)$ of k -differentiable curves from \mathbb{R} to Q . By definition, two given curves in Q , $\gamma_1(t)$ and $\gamma_2(t)$, where $t \in (-a, a)$ with $a \in \mathbb{R}$ have contact of order k at $q_0 = \gamma_1(0) = \gamma_2(0)$ if there is a local chart (φ, U) of Q such that $q_0 \in U$ and

$$\left. \frac{d^s}{dt^s} (\varphi \circ \gamma_1(t)) \right|_{t=0} = \left. \frac{d^s}{dt^s} (\varphi \circ \gamma_2(t)) \right|_{t=0},$$

for all $s = 0, \dots, k$. This is a well-defined equivalence relation in $C^k(\mathbb{R}, Q)$ and the equivalence class of a curve γ will be denoted by $[\gamma]_0^{(k)}$. The set of equivalence classes will be denoted by $T^{(k)}Q$ and it is not hard to show that it has a natural structure of differentiable manifold. Moreover, $\tau_Q^k : T^{(k)}Q \rightarrow Q$ where $\tau_Q^k([\gamma]_0^{(k)}) = \gamma(0)$ is a fiber bundle called the *tangent bundle of order k* of Q .

When there exists a group action $\phi : G \times Q \rightarrow Q$, we shall naturally define a lift to the group action $\phi^{(k)} : G \times T^{(k)}Q \rightarrow T^{(k)}Q$ given by

$$\phi_g^{(k)}([\gamma]_0^{(k)}) := [\phi_g \circ \gamma]_0^{(k)}.$$

This action endows $T^{(k)}Q$ with a principal G -bundle structure. The quotient $T^{(k)}Q/G$ is a fiber bundle over the base Q/G . The class of elements $[\gamma]_0^{(k)}$ in the quotient $(T^{(k)}Q/G)$ is denoted $[\gamma]_0^{(k)}|_G$.

Now, let G be a Lie group and consider, as before, the left-translation on itself

$$\ell : G \times G \rightarrow G, \quad (g, h) \mapsto \ell_g(h) = gh.$$

Obviously ℓ_g is a diffeomorphism (the same is valid for the right-translation, but in the sequel we only work with the left-translation, for sake of simplicity).

The left-translation allows us to trivialize the tangent bundle TG and the cotangent bundle T^*G as follows

$$\begin{aligned} TG &\rightarrow G \times \mathfrak{g}, & (g, \dot{g}) &\mapsto (g, g^{-1}\dot{g}) = (g, T_g\ell_{g^{-1}}\dot{g}) = (g, \xi), \\ T^*G &\rightarrow G \times \mathfrak{g}^*, & (g, \alpha_g) &\mapsto (g, T_e^*\ell_g(\alpha_g)) = (g, \alpha), \end{aligned}$$

where $\mathfrak{g} = T_eG$ is the Lie algebra of G and e is the neutral element of G . In the same way, we have the following identifications: $TTG \equiv G \times 3\mathfrak{g}$ (where $3\mathfrak{g}$ stands for $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$). Throughout this paper, the notation nV , where V is a given space, denotes the cartesian product of n copies of V) and $T^*TG = G \times \mathfrak{g}^* \times \mathfrak{g} \times \mathfrak{g}^*$. Therefore, in the case when the manifold Q has a Lie group structure, i.e. $Q = G$, we can also use the left trivialization to identify the higher-order tangent bundle $T^{(k)}G$ with $G \times k\mathfrak{g}$. That is, if $g : I \rightarrow G$ is a curve in $C^{(k)}(\mathbb{R}, G)$:

$$\begin{aligned} \Upsilon^{(k)} : T^{(k)}G &\longrightarrow G \times k\mathfrak{g} \\ [g]_0^{(k)} &\longmapsto (g(0), g^{-1}(0)\dot{g}(0), \left. \frac{d}{dt} \right|_{t=0} (g^{-1}(t)\dot{g}(t)), \dots, \left. \frac{d^{k-1}}{dt^{k-1}} \right|_{t=0} (g^{-1}(t)\dot{g}(t))). \end{aligned}$$

It is clear that $\Upsilon^{(k)}$ is a diffeomorphism.

We will denote $\xi(t) = g^{-1}(t)\dot{g}(t)$, and this equation shall be considered in the following as the continuous *reconstruction equation*. Therefore

$$\Upsilon^{(k)}([g]_0^{(k)}) = (g, \xi, \dot{\xi}, \dots, \xi^{(k-1)}) ,$$

where

$$\xi^{(l)}(t) = \frac{d^l}{dt^l}(g^{-1}(t)\dot{g}(t)), \quad 0 \leq l \leq k-1$$

and $g(0) = g, \xi^{(l)}(0) = \xi^{(l)}, 0 \leq l \leq k-1$. We will use the following notations without distinction $\xi^{(0)} = \xi, \xi^{(1)} = \dot{\xi}$, when referring to the derivatives.

We may also define the surjective mappings $\tau_G^{(l,k)} : T^{(k)}G \rightarrow T^{(l)}G$, for $l \leq k$, given by $\tau_G^{(l,k)}([g]_0^{(k)}) = [g]_0^{(l)}$. With the previous identifications we have that

$$\tau_G^{(l,k)}(g(0), \xi(0), \dot{\xi}(0), \dots, \xi^{(k-1)}(0)) = (g(0), \xi(0), \dot{\xi}(0), \dots, \xi^{(l-1)}(0))$$

It is easy to see that $T^{(1)}G \equiv G \times \mathfrak{g}$, $T^{(0)}G \equiv G$ and $\tau_G^{(0,k)} = \tau_G^k$.

Now, we consider the canonical immersion $j_k : T^{(k)}G \rightarrow T(T^{(k-1)}G)$ defined as $j_k([g]_0^{(k)}) = [g^{(k-1)}]_0^{(1)}$, where $g^{(k-1)}$ is the lift of the curve g to $T^{(k-1)}G$; that is, the curve $g^{(k-1)} : \mathbb{R} \rightarrow T^{(k-1)}G$ is given by $g^{(k-1)}(t) = [g_t]_0^{(k-1)}$ where $g_t(s) = g(t+s)$. Using the identification given by $\Upsilon^{(k)}$ we have that:

$$\begin{aligned} j^{(k)} : \quad G \times k\mathfrak{g} &\longrightarrow G \times (2k-1)\mathfrak{g} \\ (g, \xi, \dot{\xi}, \dots, \xi^{(k-1)}) &\longmapsto (g, \xi, \dot{\xi}, \dots, \xi^{(k-2)}; \xi, \dot{\xi}, \dots, \xi^{(k)}) \end{aligned}$$

where we identify $T(T^{(k-1)}G) \equiv T(G \times (k-1)\mathfrak{g}) \equiv G \times (2k-1)\mathfrak{g}$, in the natural way.

3.2. Euler-Lagrange equations for trivial principal bundles. Now, we derive from a variational procedure the Euler-Lagrange equations for the trivial principal bundle $Q = M \times G$ where M is a n -dimensional differentiable manifold with coordinates (q^i) , $1 \leq i \leq n$, and G a Lie group.

Let $L : TQ \rightarrow \mathbb{R}$ be a Lagrangian function. Since $TQ \simeq TM \times TG$ and $TG \simeq G \times \mathfrak{g}$ from the left-trivialization, we consider our Lagrangian function as $L : TM \times G \times \mathfrak{g} \rightarrow \mathbb{R}$.

The motion of the mechanical system is described by applying the following principle,

$$\delta \int_0^T L(q(t), \dot{q}(t), g(t), \xi(t)) dt = 0 \quad (4)$$

for all variations $\delta q(t)$ where $\delta q(0) = \delta q(T) = 0$, $q(t) \in M$ and $\delta \xi$ verifying $\delta \xi(t) = \dot{\eta}(t) + [\xi(t), \eta(t)]$, where $\eta(t)$ is an arbitrary curve on the Lie algebra with $\eta(0) = \eta(T) = 0$ and $\eta = g^{-1}\delta g$ (see [24]). This principle gives rise to the Euler-Lagrange equations on trivial principal bundles given by

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}, \quad (5a)$$

$$\frac{d}{dt} \left(\frac{\delta L}{\delta \xi} \right) = \text{ad}_\xi^* \left(\frac{\delta L}{\delta \xi} \right) + \ell_g^* \frac{\delta L}{\delta g}, \quad (5b)$$

where $\text{ad}_\xi \eta = [\xi, \eta]$. If the Lagrangian L is left-invariant the above equations are written as

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q} \quad (6a)$$

$$\frac{d}{dt} \left(\frac{\delta L}{\delta \xi} \right) = \text{ad}_\xi^* \left(\frac{\delta L}{\delta \xi} \right), \quad (6b)$$

which shall be considered as the Euler-Poincaré equations for Lagrangian systems defined on trivial principal bundles.

3.3. Second-order Euler-Lagrange equations for trivial principal bundles.

In this subsection we deduce, from a variational principle, the Euler-Lagrange equations for Lagrangians defined on $T^{(2)}Q \simeq T^{(2)}M \times G \times 2\mathfrak{g}$ from a left-trivization.

Let $L : T^{(2)}Q \simeq T^{(2)}M \times G \times 2\mathfrak{g} \rightarrow \mathbb{R}$ be a Lagrangian function, $L(q, \dot{q}, \ddot{q}, g, \dot{g}, \ddot{g}) \equiv L(q, \dot{q}, \ddot{q}, g, \xi, \dot{\xi})$ where $\xi = g^{-1}\dot{g}$. The problem consists on finding the critical curves of the action defined by

$$\mathcal{A}(c(t)) = \int_0^T L(q, \dot{q}, \ddot{q}, g, \xi, \dot{\xi}) dt$$

among all the curves $c(t) \in \mathcal{C}^\infty(T^{(2)}M \times G \times 2\mathfrak{g})$ satisfying the boundary conditions for arbitrary variations $\delta c = (\delta q, \delta q^{(1)}, \delta q^{(2)}, \delta g, \delta \xi, \delta \dot{\xi})$, where $\delta q = \frac{d}{d\epsilon}|_{\epsilon=0} q_\epsilon$, $\delta q^{(l)} = \frac{d^l}{dt^l} \delta q$, for $l = 1, 2$; and $\delta g = \frac{d}{d\epsilon}|_{\epsilon=0} g_\epsilon$. Here, $\epsilon \mapsto g_\epsilon$ and $\epsilon \mapsto q_\epsilon$ are smooth curves in G and M respectively, for $\epsilon \in (-a, a) \subset \mathbb{R}$, such that $g_0 = g$ and $q_0 = q$. We define, for any ϵ , $\xi_\epsilon := g_\epsilon^{-1} \dot{g}_\epsilon$. The corresponding variations $\delta \xi$ induced by δg are given by $\delta \xi = \dot{\eta} + [\xi, \eta]$ where $\eta := g^{-1} \delta g \in \mathfrak{g}$ ($\delta g = g\eta$). Therefore

$$\begin{aligned} \delta \mathcal{A}(c(t)) &= \delta \int_0^T L(q(t), \dot{q}(t), \ddot{q}(t), g(t), \xi(t), \dot{\xi}(t)) dt \\ &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_0^T L(q_\epsilon(t), \dot{q}_\epsilon(t), \ddot{q}_\epsilon(t), g_\epsilon(t), \xi_\epsilon(t), \dot{\xi}_\epsilon(t)) dt \\ &= \int_0^T \left(\left\langle \frac{\partial L}{\partial q}, \delta q \right\rangle + \left\langle \frac{\partial L}{\partial \dot{q}}, \frac{d}{dt}(\delta q) \right\rangle + \left\langle \frac{\partial L}{\partial \ddot{q}}, \frac{d^2}{dt^2}(\delta q) \right\rangle + \left\langle \frac{\partial L}{\partial g}, \delta g \right\rangle \right. \\ &\quad \left. + \left\langle \frac{\delta L}{\delta \xi}, \delta \xi \right\rangle + \left\langle \frac{\delta L}{\delta \dot{\xi}}, \frac{d}{dt}(\delta \xi) \right\rangle \right) dt \end{aligned}$$

Employing the integration by parts (twice) and the vanishing initial and endpoint conditions $q(0) = q(T) = \dot{q}(0) = \dot{q}(T) = 0$ and $\eta(0) = \eta(T) = \dot{\eta}(0) = \dot{\eta}(T) = 0$, the stationary condition $\delta \mathcal{A} = 0$ implies

$$\begin{aligned} &\int_0^T \left\langle \left(-\frac{d}{dt} + \text{ad}_\xi^* \right) \left(\frac{\delta L}{\delta \xi} - \frac{d}{dt} \frac{\delta L}{\delta \dot{\xi}} \right), \eta \right\rangle dt + \int_0^T \left\langle \ell_g^* \left(\frac{\partial L}{\partial g} \right), \eta \right\rangle dt \\ &+ \int_0^T \left\langle \frac{d}{dt} \left(\frac{d}{dt} \frac{\partial L}{\partial \ddot{q}} - \frac{\partial L}{\partial \dot{q}} \right) + \frac{\partial L}{\partial q}, \delta q \right\rangle dt = 0. \end{aligned}$$

Therefore, $\delta\mathcal{A}(c(t)) = 0$ if and only if $c(t) \in \mathcal{C}^\infty(T^{(2)}M \times G \times 2\mathfrak{g})$ is a solution of the Euler-Lagrange equations for $L : T^{(2)}M \times G \times 2\mathfrak{g} \rightarrow \mathbb{R}$,

$$\left(\frac{d}{dt} - \text{ad}_\xi^*\right) \left(\frac{\delta L}{\delta \xi} - \frac{d}{dt} \frac{\delta L}{\delta \dot{\xi}}\right) = \ell_g^* \frac{\partial L}{\partial g}, \quad (7a)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \ddot{q}}\right) = \frac{\partial L}{\partial q}, \quad (7b)$$

which splits into a M part (7a) and a G part (7b). From equations (7) we can obtain the following theorem

Theorem 3.1. *Let $L : T^{(2)}Q \simeq T^{(2)}M \times G \times 2\mathfrak{g} \rightarrow \mathbb{R}$ be a left-trivialized second-order Lagrangian, $\xi = g^{-1}\dot{g}$, and $\eta(t)$ a curve on \mathfrak{g} with fixed endpoints $\eta(0) = \eta(T) = 0$. The curve $c(t) \in \mathcal{C}^\infty(T^{(2)}M \times G \times 2\mathfrak{g})$ satisfies $\delta\mathcal{A}(c(t)) = 0$ for the action $\mathcal{A} : \mathcal{C}^\infty(T^{(2)}M \times G \times 2\mathfrak{g}) \rightarrow \mathbb{R}$ given by*

$$\mathcal{A}(c(t)) = \int_0^T L(q, \dot{q}, \ddot{q}, g, \xi, \dot{\xi}) dt,$$

with respect to the variations $\delta q, \delta q^{(l)} = \frac{d^l}{dt^l} \delta q$ for $l = 1, 2$ (such that $\delta q(0) = \delta q(T) = 0$ and $\delta \dot{q}(0) = \delta \dot{q}(T) = 0$); δg and $\delta \xi = \dot{\eta} + \text{ad}_\xi \eta$, if and only if $c(t)$ is a solution of the Euler-Lagrange equations,

$$\begin{aligned} \ell_g^* \frac{\partial L}{\partial g} - \frac{d}{dt} \frac{\delta L}{\delta \xi} + \frac{d^2}{dt^2} \frac{\delta L}{\delta \dot{\xi}} + \text{ad}_\xi^* \frac{\delta L}{\delta \xi} - \text{ad}_\xi^* \left(\frac{d}{dt} \frac{\delta L}{\delta \dot{\xi}} \right) &= 0, \\ \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q} &= 0, \end{aligned}$$

which are just the expansion of (7).

Corollary 3.1. *If the Lagrangian is left-invariant, that is if L does not depend of $g \in G$, the equations of motion are given by*

$$\frac{d^2}{dt^2} \frac{\delta L}{\delta \dot{\xi}} - \frac{d}{dt} \frac{\delta L}{\delta \xi} + \text{ad}_\xi^* \frac{\delta L}{\delta \xi} - \text{ad}_\xi^* \left(\frac{d}{dt} \frac{\delta L}{\delta \dot{\xi}} \right) = 0, \quad (8a)$$

$$\frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q} = 0, \quad (8b)$$

which shall be considered the Euler-Poincaré equations for second order trivial principal bundles.

Remark 3.2. Higher-order Euler-Lagrange equations on trivial principal bundles. The previous setting can be extended to Lagrangians defined on a higher-order tangent bundle of a configuration manifold given by a trivial principal bundle. We identify the higher-order tangent bundle $T^{(k)}Q = T^{(k)}(M \times G) \simeq T^{(k)}M \times G \times k\mathfrak{g}$.

Let L be a higher-order Lagrangian defined on $T^{(k)}M \times G \times k\mathfrak{g}$ where we have local coordinates $(q, \dot{q}, \ddot{q}, \dots, q^{(k)}, g, \xi, \dot{\xi}, \dots, \xi^{(k-1)})$, $\xi = g^{-1}\dot{g}$. Let us denote the

variations

$$\delta q = \frac{d}{d\epsilon} \Big|_{\epsilon=0} q_\epsilon, \quad \delta q^{(l)} = \left(\frac{d^l}{dt^l} \right) \delta q, \quad \delta \xi^{(l)} = \frac{d^j}{dt^j} (\delta \xi), \quad \delta g = \frac{d}{d\epsilon} \Big|_{\epsilon=0} g_\epsilon$$

for $l = 1, \dots, k$; $j = 1, \dots, k-1$ and where the variation $\delta \xi$ is induced by δg as $\delta \xi = \dot{\eta} + [\xi, \eta]$ where η is a curve on the Lie algebra with fixed endpoints. Therefore, we can deduce that from Hamilton's principle, integrating k times by parts and using the boundary conditions

$$\begin{aligned} \delta q(0) &= \delta q(T) = 0, \\ \delta q^{(j)}(0) &= \delta q^{(j)}(T) = 0, \quad j = 1, \dots, k \\ \eta(0) &= \dot{\eta}(0) = \dots = \eta^{(k-1)}(0) = 0, \\ \eta(T) &= \dot{\eta}(T) = \dots = \eta^{(k-1)}(T) = 0, \end{aligned}$$

(and therefore, $\delta \xi^{(l)}(0) = \delta \xi^{(l)}(T) = 0$, for $l = 1, \dots, k$) the higher-order Euler-Lagrange equations for the Lagrangian $L : T^{(k)}M \times G \times k\mathfrak{g} \rightarrow \mathbb{R}$ are

$$\begin{aligned} \sum_{l=0}^k (-1)^l \frac{d^l}{dt^l} \left(\frac{\partial L}{\partial q^{(l)i}} \right) &= 0, \\ \left(\frac{d}{dt} - \text{ad}_\xi^* \right) \sum_{l=0}^{k-1} (-1)^l \frac{d^l}{dt^l} \left(\frac{\partial L}{\partial \xi^{(l)}} \right) &= \ell_g^* \left(\frac{\partial L}{\partial g} \right). \end{aligned}$$

As in the previous cases, if the Lagrangian is left-invariant the right hand side of the second equation vanishes.

4. DISCRETE HIGHER-ORDER EULER-LAGRANGE EQUATIONS ON TRIVIAL PRINCIPAL BUNDLES

In this section we derive, from a discrete variational point of view, the discrete Euler-Lagrange equations for Lagrangians defined on a higher-order tangent bundle to $Q = M \times G$, that is $T^{(k)}M \times G \times k\mathfrak{g}$, where G is a finite dimensional Lie group and \mathfrak{g} its Lie algebra. This variational procedure gives rise to obtain a variational integrator. As an application, we will consider the optimal control of mechanical systems.

4.1. Discrete second-order Euler Lagrange equations on trivial principal bundles. Let G be a finite dimensional Lie group and consider the associated discrete problem. We chose $3(M \times G) \equiv 3M \times 3G$ as the natural discretization of $T^{(2)}Q$ since we are considering the left trivialization $T^{(2)}G \simeq G \times 2\mathfrak{g}$. Therefore, we develop the discrete Euler-Lagrange equations for a discrete Lagrangian $L_d : 3(M \times G) \rightarrow \mathbb{R}$. We define the *reconstruction equation* $W_k = g_k^{-1}g_{k+1}$. Taking variations and considering $\Sigma_k = g_k^{-1}\delta g_k$ we obtain

$$\delta W_k = -\Sigma_k W_k + W_k \Sigma_{k+1}, \tag{9}$$

where $g_k, W_k \in G$ and $\Sigma_k \in \mathfrak{g}$.

Definition 4.1 (Discrete Hamilton's principle for trivial principal bundles). Let define the space of sequences $\mathcal{C}^{2(N+1)} = \{(q_0, q_1, \dots, q_N, g_0, g_1, \dots, g_N) \in (N+1)M \times (N+1)G\}$, assuming that (q_0, g_0) , (q_1, g_1) , (q_{N-1}, g_{N-1}) and (q_N, g_N) are fixed, we define the discrete action associated with the Lagrangian $L_d : 3(M \times G) \rightarrow \mathbb{R}$ as

$$\mathcal{A}_d(q_{0:N}, g_{0:N-2}, W_{0:N-1}) = \sum_{k=0}^{N-2} L_d(q_k, q_{k+1}, q_{k+2}, g_k, W_k, W_{k+1}). \quad (10)$$

Hamilton's principle establishes that the sequence $(q_0, q_1, \dots, q_N, g_0, g_1, \dots, g_N) \in \mathcal{C}^{2(N+1)}$ is a solution of the Lagrangian system determined by $L_d : 3(M \times G) \rightarrow \mathbb{R}$ if and only if $(q_0, q_1, \dots, q_N, g_0, g_1, \dots, g_N)$ is a critical point of \mathcal{A}_d .

The equations of motion are the critical paths of the discrete action, then, $\delta \mathcal{A}_d = 0$ if and only if

$$\begin{aligned} 0 &= \delta \sum_{k=0}^{N-2} L_d(q_k, q_{k+1}, q_{k+2}, g_k, W_k, W_{k+1}) \\ &= \sum_{k=0}^{N-2} \left(\sum_{j=1}^3 (D_j L_d) \delta q_{j+k-1} + (D_4 L_d) \delta g_k + \sum_{j=5}^6 (D_j L_d) \delta W_{j+k-5} \right). \end{aligned} \quad (11)$$

Here, D_l denotes the partial derivative with respect to the l -th variable. Rearranging the sum indexes, (11) can be decomposed in the following way:

$$\begin{aligned} &\sum_{k=0}^{N-2} \sum_{j=1}^3 (D_j L_d) \delta q_{j+k-1} = \\ &\sum_{k=0}^{N-2} (D_1 L_d(q_k, q_{k+1}, q_{k+2}, g_k, W_k, W_{k+1}) + D_2 L_d(q_{k-1}, q_k, q_{k+1}, g_{k-1}, W_{k-1}, W_k) \\ &+ D_3 L_d(q_{k-2}, q_{k-1}, q_k, g_{k-2}, W_{k-2}, W_{k-1})) \delta q_k, \\ &\sum_{k=0}^{N-2} (D_4 L_d) \delta g_k + \sum_{k=0}^{N-2} \sum_{j=5}^6 (D_j L_d) \delta W_{j+k-5} = \\ &\sum_{k=0}^{N-2} D_1 L_d(g_k, W_k, W_{k+1}) \delta g_k + \sum_{k=0}^{N-2} \sum_{j=2}^3 D_j L_d(g_k, W_k, W_{k+1}) \delta W_{j+k-2} = \\ &\sum_{k=0}^{N-2} D_1 L_d(g_k, W_k, W_{k+1}) (g_k \Sigma_k) \\ &+ \sum_{k=0}^{N-2} \sum_{j=2}^3 D_j L_d(g_k, W_k, W_{k+1}) (-\Sigma_{j+k-2} W_{j+k-2} + W_{j+k-2} \Sigma_{j+k-1}), \end{aligned}$$

where we have used the relations $\Sigma_k = g_k^{-1} \delta g_k$ and (9). Moreover, from the second row we have considered that the M variables (q_k, q_{k+1}, q_{k+2}) are fixed. From these equalities we obtain the following theorem:

Theorem 4.1. *Let $L : T^{(2)}Q \rightarrow \mathbb{R}$ be a Lagrangian where $Q = M \times G$ and $T^{(2)}Q$ is left-trivialized as $T^{(2)}Q \simeq T^{(2)}M \times G \times 2\mathfrak{g}$. Let us consider $c \in \mathcal{C}^{2(N+1)}$. Then, the path c satisfies $\delta\mathcal{A}_d = 0$ for $\mathcal{A}_d(c(t)) : \mathcal{C}^{2(N+1)} \rightarrow \mathbb{R}$, given in definition (4.1) with respect to arbitrary variations $\delta q_k, \delta g_k, \delta W_k$, also satisfying $(q_0, \Sigma_0), (q_1, \Sigma_1), (q_{N-1}, \Sigma_{N-1}), (q_N, \Sigma_N)$ fixed, $\Sigma_k = g_k^{-1}\delta g_k$, if and only if $c(t)$ satisfies the discrete Euler-Lagrange equations*

$$0 = D_1 L_d|_g(q_k, q_{k+1}, q_{k+2}) + D_2 L_d|_g(q_{k-1}, q_k, q_{k+1}) + D_3 L_d|_g(q_{k-2}, q_{k+1}, q_k),$$

$$\begin{aligned} 0 &= \ell_{g_{k-1}}^* D_1 L_d|_q(g_{k-1}, W_{k-1}, W_k) \\ &+ \ell_{W_{k-1}}^* D_2 L_d|_q(g_{k-1}, W_{k-1}, W_k) - r_{W_k}^* D_2 L_d|_q(g_k, W_k, W_{k+1}) \\ &- r_{W_k}^* D_3 L_d|_q(g_{k-1}, W_{k-1}, W_k) + \ell_{W_{k-1}}^* D_3 L_d|_q(g_{k-2}, W_{k-2}, W_{k-1}), \end{aligned}$$

$$W_k = g_k^{-1} g_{k+1},$$

for $k = 2, \dots, N-2$.

The notation $L_d|_g$ implies that the M variables are frozen while, correspondingly, $L_d|_q$ implies that the G variables are frozen.

Corollary 4.2. *If L_d is left-invariant then we can define the reduced Lagrangian $\tilde{L}_d : 3M \times 2G \rightarrow \mathbb{R}$ and the equations in theorem 4.1 are rewritten as*

$$0 = D_1 \tilde{L}_d|_g(q_k, q_{k+1}, q_{k+2}) + D_2 \tilde{L}_d|_g(q_{k-1}, q_k, q_{k+1}) + D_3 \tilde{L}_d|_g(q_{k-2}, q_{k+1}, q_k),$$

$$\begin{aligned} 0 &= \ell_{W_{k-1}}^* D_1 \tilde{L}_d|_q(W_{k-1}, W_k) - r_{W_k}^* D_1 \tilde{L}_d|_q(W_k, W_{k+1}) \\ &- r_{W_k}^* D_2 \tilde{L}_d|_q(W_{k-1}, W_k) + \ell_{W_{k-1}}^* D_2 \tilde{L}_d|_q(W_{k-2}, W_{k-1}), \end{aligned}$$

$$W_k = g_k^{-1} g_{k+1},$$

for $k = 2, \dots, N-2$. These should be consider as the discrete Euler-Poincaré equations for second order trivial principal bundles.

Remark 4.3 (Discrete higher-order Euler-Lagrange equations). Is easy to extend these techniques for higher order discrete mechanics. Consider a mechanical system determined by a Lagrangian $L : T^{(k)}(M \times G) \rightarrow \mathbb{R}$. It is well known that the tangent bundle $T^{(k)}(M \times G)$ can be left-trivialized as $T^{(k)}(M \times G) \simeq T^{(k)}M \times G \times k\mathfrak{g}$, where \mathfrak{g} is the Lie algebra G .

Now, we consider the associated discrete problem by replacing the higher order tangent bundle by $(k+1)$ copies of the manifold and the group. At this point, we develop the discrete Euler-Lagrange equations for the discrete Lagrangians defined on $(k+1)(M \times G) \equiv (k+1)M \times (k+1)G$.

Let $L_d : (k+1)(M \times G) \rightarrow \mathbb{R}$ be a discrete Lagrangian where G is a finite dimensional Lie group and M a n -dimensional differentiable manifold. As before, denote by $W_i = g_i^{-1} g_{i+1}$ and $\Sigma_i = g_i^{-1} \delta g_i$. Taking variations over W_i we obtain

$\delta W_i = -\Sigma_i W_i + W_i \Sigma_{i+1}$, where $g_i, W_i \in G$ and $\Sigma_i \in \mathfrak{g}$. By Hamilton's principle the equations of motion are the critical paths of the discrete action

$$\sum_{i=0}^{N-k} L_d(q_{(i,i+k)}, g_i, W_{(i,i+k-1)})$$

with boundary conditions $q_{(0,k-1)} = q_{(N-k+1,N)} = 0$, $\Sigma_{(0,k-1)} = 0$, $\Sigma_{(N-k+1,N)} = 0$ and g_0, \dots, g_{k-1} and g_{N-k+1}, \dots, g_N fixed, where we employ the notation

$$\begin{aligned} q_{(i,j)} &= (q_i, q_{i+1}, \dots, q_{j-1}, q_j), \\ W_{(i,j)} &= (W_i, W_{i+1}, \dots, W_{j-1}, W_j), \\ \Sigma_{(i,j)} &= (\Sigma_i, \Sigma_{i+1}, \dots, \Sigma_j). \end{aligned}$$

Taking variations we deduce that,

$$\begin{aligned} &\delta \sum_{i=0}^{N-k} L_d(q_{(i,i+k)}, g_i, W_{(i,i+k-1)}) = \\ &\sum_{i=k}^{N-k} \left[D_1 L_d|_q(g_i, W_{(i,i+k)}) (g_i \Sigma_i) + \sum_{j=1}^{k+1} D_j L_d|_g(q_{(i-j+1, i-j+1+k)}) \delta q_i \right. \\ &\quad \left. + \sum_{j=2}^{k+1} D_j L_d|_q(g_i, W_{(i,i+k-1)}) (-\Sigma_{j+i-2} W_{j+i-2} + W_{j+i-2} \Sigma_{j+i-1}) \right], \end{aligned}$$

where again we have used the relation (9). Therefore, the *discrete higher-order Euler-Lagrange equations* on $(k+1)(M \times G)$ are given by

$$\begin{aligned} 0 &= \sum_{j=1}^{k+1} D_j L_d|_g(q_{(i-j+1, i-j+1+k)}), \\ 0 &= \ell_{g_{i-1}}^* D_1 L_d|_q(g_{i-1}, W_{(i-1, i+k-1)}) \\ &\quad + \sum_{j=2}^{k+1} \left(\ell_{W_{i-1}}^* \right) D_j L_d|_q(g_{i-j+1}, W_{(i-j+1, i-j+k)}) \\ &\quad - \sum_{j=2}^{k+1} \left(r_{W_i}^* \right) D_j L_d|_q(g_{i-j+2}, W_{(i-j+2, i-j+k+1)}), \end{aligned}$$

where $k \leq i \leq N-k$. These equations, together with the reconstruction equation $W_i = g_i^{-1} g_{i+1}$ are called *discrete higher-order Euler-Lagrange equations* on $(k+1)(M \times G)$.

Finally, if L_d is left-invariant we will define the reduced Lagrangian $\tilde{L}_d : (k+1)M \times kG \rightarrow \mathbb{R}$. Then the *discrete higher-order Euler-Poincaré equations* on the reduced space $(k+1)M \times kG$ are given by

$$\begin{aligned}
0 &= \sum_{j=1}^{k+1} D_j \tilde{L}_d|_g(q_{(i-j+1, i-j+1+k)}), \\
0 &= \sum_{j=2}^{k+1} \left(\ell_{W_{i-1}}^* \right) D_j \tilde{L}_d|_q(W_{(i-j+1, i-j+k)}) \\
&\quad - \sum_{j=2}^{k+1} \left(r_{W_i}^* \right) D_j \tilde{L}_d|_q(W_{(i-j+2, i-j+k+1)}), \\
W_i &= g_i^{-1} g_{i+1},
\end{aligned}$$

for $k \leq i \leq N - k$.

5. DISCRETE AND CONTINUOUS MECHANICAL SYSTEMS WITH CONSTRAINTS ON HIGHER-ORDER TRIVIAL PRINCIPAL BUNDLES

In this section we derive, from a discrete variational principle, an integrator for higher-order mechanics with higher-order constraints. As an application, we employ the presented techniques to solve optimal control problems for underactuated systems.

5.1. Mechanical systems defined on higher-order trivial principal bundles subject to constraints. Consider a higher-order Lagrangian systems given by $L : T^{(k)}M \times G \times k\mathfrak{g} \rightarrow \mathbb{R}$ with higher-order constraints given by $\Phi^\alpha : T^{(k)}M \times G \times k\mathfrak{g} \rightarrow \mathbb{R}$, $1 \leq \alpha \leq m$. We denote by $\tilde{\mathcal{M}}$ the constraint submanifold locally determined by the vanishing of these m constraints. Define the action sum

$$\mathcal{A}(c(t)) = \int_0^T L(c(t)) dt,$$

where $c(t)$ is a curve in $T^{(k)}M \times G \times k\mathfrak{g}$ with local coordinates

$$c(t) = (q(t), \dot{q}(t), \dots, q^{(k)}(t), g(t), \xi(t), \dot{\xi}(t), \dots, \xi^{(k-1)}(t)). \quad (12)$$

The pure variational principle for this kind of higher-order mechanical systems is given by

$$\begin{cases} \min \mathcal{A}(c(t)) \\ \text{subject to } \Phi^\alpha(c(t)) = 0, \quad 1 \leq \alpha \leq m \end{cases}$$

Moreover, we shall consider the boundary conditions $q(0) = q(T) = q^{(l)}(0) = q^{(l)}(T) = 0$ for $l = 1, \dots, k$; $\eta^{(j)}(0) = \eta^{(j)}(T) = 0$ for $j = 0, \dots, k-1$; and $\xi = g^{-1}\dot{g}$. Here, $\eta(t)$ is a curve in the Lie algebra \mathfrak{g} with fixed endpoints induced by the variations $\delta\xi = \dot{\eta} + [\xi, \eta]$.

Definition 5.1. A curve $c(t) \in \mathcal{C}^\infty(T^{(k)}M \times G \times k\mathfrak{g})$ will be called a solution of the higher-order variational problem with constraints if c is a critical point of $\mathcal{A}|_{\tilde{\mathcal{M}}}$.

As in the case of systems defined in a tangent bundle TM , and subject to constraints on the same tangent bundle, by using the Lagrange multipliers theorem we may characterize the regular critical points of the higher-order problem with

constraints as an unconstrained problem for an extended Lagrangian system. (See [32] for a detailed proof.)

Proposition 5.1 (Variational problem with higher-order constraints). *A curve $c \in \mathcal{C}^\infty(T^{(k)}M \times G \times k\mathfrak{g})$ is a critical point of the variational problem with higher-order constraints if and only if c is a critical point of the functional*

$$\int_0^T \tilde{L}(c(t), \lambda(t)) dt,$$

where $\lambda = (\lambda_1, \dots, \lambda_m)$ are regarded as generalized coordinates on \mathbb{R}^m . Moreover, $\tilde{L} : T^{(k)}M \times G \times k\mathfrak{g} \times \mathbb{R}^m \rightarrow \mathbb{R}$ is defined by

$$\tilde{L}(c(t), \lambda) = L(c(t)) - \lambda_\alpha \Phi^\alpha(c(t)).$$

The equations of motion given by the higher-order variational principle with constraints read:

$$\begin{aligned} 0 &= \sum_{l=0}^k (-1)^l \frac{d^l}{dt^l} \left(\frac{\partial L}{\partial q^{(l)}} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q^{(l)}} \right), \\ 0 &= \left(\frac{d}{dt} - \text{ad}_\xi^* \right) \sum_{l=0}^{k-1} (-1)^l \frac{d^l}{dt^l} \left(\frac{\partial L}{\partial \xi^{(l)}} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial \xi^{(l)}} \right) - \ell_g^* \left(\frac{\partial L}{\partial g} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial g} \right), \\ 0 &= \Phi^\alpha(c(t)), \\ \dot{g} &= g\xi, \end{aligned}$$

for $1 \leq \alpha \leq m$. These are called the Euler-Lagrange equations with higher-order constraints on $T^{(k)}Q \times \mathbb{R}^m$, where $Q = M \times G$.

If the Lagrangian is left-invariant (that is, L does not depend of the variables on G) these equations are rewritten as the higher-order Euler-Lagrange equations with higher-order constraints on $T^{(k)}M \times k\mathfrak{g} \times \mathbb{R}^m$,

$$\begin{aligned} 0 &= \sum_{l=0}^k (-1)^l \frac{d^l}{dt^l} \left(\frac{\partial L}{\partial q^{(l)}} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial q^{(l)}} \right), \\ 0 &= \left(\frac{d}{dt} - \text{ad}_\xi^* \right) \sum_{l=0}^{k-1} (-1)^l \frac{d^l}{dt^l} \left(\frac{\partial L}{\partial \xi^{(l)}} - \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial \xi^{(l)}} \right), \\ 0 &= \Phi^\alpha(c(t)), \\ \dot{g} &= g\xi, \end{aligned}$$

for $1 \leq \alpha \leq m$. These shall be considered the Euler-Poincaré equations with higher-order constraints on $T^{(k)}Q \times \mathbb{R}^m$, where $Q = M \times G$.

5.2. Discrete variational problem with constraints on higher-order trivial principal bundles. In this subsection we consider the discrete version of higher-order Lagrangian systems subject to higher-order constraints. These elements are denoted, respectively, by $L_d : (k+1)(M \times G) \rightarrow \mathbb{R}$ and $\Phi_d^\alpha : (k+1)(M \times G) \rightarrow \mathbb{R}$, for $1 \leq \alpha \leq m$. Again, to define the discrete problem we take the Lie group G as the discrete version of the Lie algebra \mathfrak{g} . We denote by $\tilde{\mathcal{M}}_d$ the constraints

submanifold locally determined by the vanishing of these m constraints. In the discrete case, the curve (12) is replaced by the sequence

$$\{q_{(i,i+k)}, g_i, W_{(i,i+k-1)}\}, \quad (13)$$

for $i = 1, \dots, N - k$. Let define the discrete action sum by

$$\mathcal{A}_d(q_{0:N}, g_{0:N-k}, W_{0:N-1}) = \sum_{i=0}^{N-k} L_d(q_{(i,i+k)}, g_i, W_{(i,i+k-1)}).$$

Therefore, we can consider the following problem as *higher-order discrete variational calculus with constraints*:

$$\begin{cases} \min \mathcal{A}_d \\ \text{subject to : } \Phi_d^\alpha(q_{(i,i+k)}, g_i, W_{(i,i+k-1)}) = 0, \end{cases}$$

where $q_{(0,k-1)}$, $q_{(N-k+1,N)}$, $g_{(0,k-1)}$, $g_{(N-k+1,N)}$ are fixed, $W_i = g_i^{-1}g_{i+1}$ and the indices have the following range: $\alpha = 1, \dots, m$, $i = 0, \dots, N - k$.

It is well-know that this classical optimization problem with higher-order constraints is equivalent to the following unconstrained higher-order variational problem for

$$\tilde{L}_d(q_{(i,i+k)}, g_i, W_{(i,i+k-1)}, \lambda_\alpha^i) = L_d(q_{(i,i+k)}, g_i, W_{(i,i+k-1)}) + \lambda_\alpha^i \Phi_d^\alpha(q_{(i,i+k)}, g_i, W_{(i,i+k-1)})$$

defined on $(k+1)(M \times G) \times \mathbb{R}^m$ with $q_{(i,i+k)} \in (k+1)M$, $g_i \in G$, $\lambda_\alpha = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$, and $W_{(i,i+k-1)} \in kG$ with $i = 0, \dots, N - k$. Consider the discrete action sum

$$\tilde{\mathcal{A}}_d(q_{0:N}, g_{0:N-k}, W_{0:N-1}, \lambda^{0:N-k}) = \sum_{i=0}^{N-k} \tilde{L}_d(q_{(i,i+k)}, g_i, W_{(i,i+k-1)}, \lambda_\alpha^i),$$

where $\lambda^{(0,N-k)} = (\lambda^0, \dots, \lambda^{N-k})$, and λ^i is a vector with components λ_α^i , $1 \leq \alpha \leq m$. The unconstrained higher-order variational problem is defined as $\min \tilde{\mathcal{A}}_d$ where $q_{(0,k-1)}$, $g_{(0,k-1)}$, $q_{(N-k+1,N)}$, $g_{(N-k+1,N)}$ are fixed and $i = 0, \dots, N - k$. The critical points of the unconstrained problem, will be those annihilating $\partial \tilde{\mathcal{A}}_d / \partial q_i$, and the constraints equations $\partial \tilde{\mathcal{A}}_d / \partial \lambda_\alpha^i$. Thus, the *higher-order discrete Euler-Lagrange equations with constraints* on $(k+1)(M \times G)$ are

$$\begin{aligned}
0 &= \ell_{g_{i-1}}^* \left(D_1 L_d|_q(g_{i-1}, W_{(i-1, i+k-1)}) + \lambda_\alpha^{i-1} D_1 \Phi_d^\alpha|_q(g_{i-1}, W_{(i-1, i+k-1)}) \right) \\
&\quad + \sum_{j=2}^{k+1} \left(\ell_{W_{i-1}}^* \right) \left(D_j L_d|_q(g_{i-j+1}, W_{(i-j+1, i-j+k)}) \right. \\
&\quad \left. + \lambda_\alpha^{i-j+1} D_j \Phi_d^\alpha|_q(g_{i-j+1}, W_{(i-j+1, i-j+k)}) \right) \\
&\quad - \sum_{j=2}^{k+1} \left(r_{W_i}^* \right) \left(D_j L_d|_q(g_{i-j+2}, W_{(i-j+2, i-j+k+1)}) \right. \\
&\quad \left. + \lambda_\alpha^{i-j+2} D_j \Phi_d^\alpha|_q(g_{i-j+2}, W_{(i-j+2, i-j+k+1)}) \right) \\
0 &= \sum_{j=1}^{k+1} D_j L_d|_g(q_{(i-j+1, i-j+1+k)}) + \lambda_\alpha^{i-j+1} D_j \Phi_d^\alpha|_g(q_{(i-j+1, i-j+1+k)}), \\
i &= k, \dots, N-k, \\
0 &= \Phi_d^\alpha(q_{(i, i+k)}, g_i, W_{(i, i+k-1)}), \\
W_i &= g_i^{-1} g_{i+1} \quad \text{with } 0 \leq i \leq N-k,
\end{aligned}$$

with $q_{(0, k-1)}$, $q_{(N-k+1, N)}$, $g_{(0, k-1)}$, $g_{(N-k+1, N)}$ fixed points on M and G respectively.

5.3. Application to optimal control of underactuated mechanical systems. The proposal of this subsection is to study optimal control problems in the case of underactuated mechanical systems, that is, a Lagrangian control system such that the number of the control inputs is less than the dimension of the configuration space (“superarticulated mechanical system” following the nomenclature by [3]).

We shall consider the configuration space Q as the trivial principal bundle $Q = M \times G$, where, as in the previous section, G is a Lie group and M is a n -dimensional differentiable manifold. In what follows we assume that all the control systems are controllable, that is, for any two points x_0 and x_f in the configuration space Q , there exists an admissible control $u(t)$ defined on some interval $[0, T]$ such that the system with initial condition x_0 reaches the point x_f in time T (see [4, 9] for more details).

Define the control manifold $U \subset \mathbb{R}^r$ where $u(t) \in U$ is the control parameter. Consider the left-trivialized Lagrangian $L : TQ \simeq TM \times \mathfrak{g} \rightarrow \mathbb{R}$, (where \mathfrak{g} is the Lie algebra associated to the Lie group G). The equations of motion of the system shall be considered the *controlled Euler-Lagrange equations*

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} = u_a \mu_A^a(q), \quad (14a)$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \xi} \right) - \text{ad}_\xi^* \left(\frac{\partial L}{\partial \xi} \right) = u_a \eta_i^a(q), \quad (14b)$$

where we denote by $\mathcal{B}^a = \{(\mu^a, \eta^a)\}$, $\mu^a(q) \in T_q^*M$, $\eta^a(q) \in \mathfrak{g}^*$, $a = 1, \dots, r$; and $A = 1, \dots, n$. Here, we are assuming that $\{(\mu^a, \eta^a)\}$ are independent elements of $\Gamma(T^*M \times \mathfrak{g}^*)$ and u_a are admissible controls. Taking this into account, the optimal control problem can be formulated as: finding trajectories $(q(t), \xi(t), u(t))$ of the state variables and control inputs satisfying (14), subject to initial conditions $(q(0), \dot{q}(0), \xi(0))$ and final conditions $(q(T), \dot{q}(T), \xi(T))$, and, extremizing the functional

$$\mathcal{J}(q, \dot{q}, \xi, u) = \int_0^T C(q(t), \dot{q}(t), \xi(t), u(t)) dt. \quad (15)$$

We can reformulate this optimal control problem as a higher-order order variational problem subject to higher-order constraints by the following procedure: complete \mathcal{B}^a to a basis $\{\mathcal{B}^a, \mathcal{B}^\alpha\}$ of the vector space $T^*M \times \mathfrak{g}^*$. Take its dual basis $\{\mathcal{B}_a, \mathcal{B}_\alpha\}$ on $\Gamma(TM \times \mathfrak{g}) = \mathfrak{X}(M) \times \mathcal{C}^\infty(M, \mathfrak{g})$. This basis induces coordinates $(q^A, \dot{q}^A, \xi^a, \xi^\alpha)$ on $TM \times \mathfrak{g}$. If we denote by $\mathcal{B}_a = \{(X_a, \chi_a)\} \in \Gamma(TM \times \mathfrak{g})$ (resp. $\mathcal{B}_\alpha = \{(X_\alpha, \chi_\alpha)\} \in \Gamma(TM \times \mathfrak{g})$), where $X_a, X_\alpha \in \mathfrak{X}(M)$; $X_a = X_a^A(q) \frac{\partial}{\partial q^A}$; $X_\alpha = X_\alpha^A(q) \frac{\partial}{\partial q^A}$ and $\chi_a(q); \chi_\alpha(q) \in \mathfrak{g}, q \in M$ then equations (14) are rewritten as

$$\left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} \right) X_a^A(q) + \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \xi} \right) - \left(\text{ad}_\xi^* \frac{\partial L}{\partial \xi} \right) \right) \chi_a(q) = u_a, \quad (16a)$$

$$\left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} \right) X_\alpha^A(q) + \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \xi} \right) - \left(\text{ad}_\xi^* \frac{\partial L}{\partial \xi} \right) \right) \chi_\alpha(q) = 0. \quad (16b)$$

As mentioned before, the proposed optimal control problem is equivalent to a variational problem with second order constraints (see [4] and reference therein), where we define the Lagrangian $\tilde{L} : T^{(2)}M \times 2\mathfrak{g} \rightarrow \mathbb{R}$ given, in the selected coordinates, by

$$\tilde{L}(q^A, \dot{q}^A, \ddot{q}^A, \xi^i, \dot{\xi}^i) = C \left(q^A, \dot{q}^A, \xi^i, F_a(q^A, \dot{q}^A, \ddot{q}^A, \xi^i, \dot{\xi}^i) \right),$$

where C is the cost function considered in (15) and

$$F_a(q^A, \dot{q}^A, \ddot{q}^A, \xi^i, \dot{\xi}^i) = \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} \right) X_a^A(q) + \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \xi} \right) - \left(\text{ad}_\xi^* \frac{\partial L}{\partial \xi} \right) \right) \chi_a(q).$$

The Lagrangian \tilde{L} is subjected to the second-order constraints:

$$\Phi^\alpha(q^A, \dot{q}^A, \ddot{q}^A, \xi^i, \dot{\xi}^i) = \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} \right) X_\alpha^A(q) + \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \xi} \right) - \left(\text{ad}_\xi^* \frac{\partial L}{\partial \xi} \right) \right) \chi_\alpha(q).$$

Thus, this kind of problems fits in the setting introduced in §5.1 by considering $k = 2$ and left invariance with respect to the group G and is illustrated in the subsequent subsections.

Remark 5.2. It is possible to extend our analysis to systems with external forces f given by the following diagram

$$\begin{array}{ccc} TM \times \mathfrak{g} & \xrightarrow{f} & T^*M \times \mathfrak{g}^* \\ & \searrow & \swarrow \\ & M & \end{array}$$

just by adding the corresponding terms in the right hand side of (14). These equations are therefore rewritten as

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial q^A} &= u_a \mu_A^a(q) + f_A(q, \dot{q}, \xi), \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \xi} \right) - \text{ad}_\xi^* \left(\frac{\partial L}{\partial \xi} \right) &= u_a \eta_i^a(q) + \bar{f}_i(q, \dot{q}, \xi), \end{aligned}$$

where

$$\begin{aligned} f : TM \times \mathfrak{g} &\longrightarrow T^*M \times \mathfrak{g}^* \\ (q, \dot{q}, \xi) &\longmapsto (f(q, \dot{q}, \xi), \bar{f}(q, \dot{q}, \xi)), \end{aligned}$$

such that $f(q, \dot{q}, \xi) = f_A(q, \dot{q}, \xi) dq^A$ and $\bar{f}(q, \dot{q}, \xi) \in \mathfrak{g}^*$.

5.3.1. Optimal control of an underactuated vehicle. Consider a rigid body moving in $SE(2)$ with a thruster to adjust its pose. The configuration of this system is determined by a tuple (x, y, θ, γ) , where (x, y) is the position of the center of mass, θ is the orientation of the blimp with respect to a fixed basis, and γ the orientation of the thrust with respect to a body basis. Therefore, the configuration manifold is $Q = SE(2) \times S^1$ (see [9] and references therein), where (x, y, θ) are the local coordinates of $SE(2)$ and γ is the local coordinate of S^1 .

The Lagrangian of the system is given by its kinetic energy

$$L(x, y, \theta, \gamma, \dot{x}, \dot{y}, \dot{\theta}, \dot{\gamma}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} J_1 \dot{\theta}^2 + \frac{1}{2} J_2 (\dot{\theta} + \dot{\gamma})^2,$$

and the input forces are

$$\begin{aligned} F^1 &= \cos(\theta + \gamma) dx + \sin(\theta + \gamma) dy - p \sin \gamma d\theta, \\ F^2 &= d\gamma, \end{aligned}$$

where the control forces that we consider are applied to a point on the body with distance $p > 0$ from the center of mass (m is the mass of the rigid body), along the body x -axis. Note this system is an example of underactuated mechanical system when the configuration space is a trivial principal bundle.

The system is invariant under the left multiplication of the Lie group $G = SE(2)$:

$$\begin{aligned} \Phi : SE(2) \times SE(2) \times S^1 &\longrightarrow SE(2) \times S^1 \\ ((a, b, \alpha), (x, y, \theta, \gamma)) &\longmapsto (x \cos \alpha - y \sin \alpha + a, x \sin \alpha + y \cos \alpha + b, \theta + \alpha, \gamma). \end{aligned}$$

A basis of the Lie algebra $\mathfrak{se}(2) \cong \mathbb{R}^3$ of $SE(2)$ is given by

$$e_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

from we have that

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = -e_2, \quad [e_2, e_3] = 0.$$

Thus, we can write down the structure constants as

$$\mathcal{C}_{31}^2 = \mathcal{C}_{23}^1 = -1, \mathcal{C}_{13}^2 = \mathcal{C}_{32}^1 = 1$$

and all others zero. An element $\xi \in \mathfrak{se}(2)$ is of the form $\xi = \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3$; therefore the reduced Lagrangian $l : TS^1 \times \mathfrak{se}(2) \rightarrow \mathbb{R}$ is given by

$$l(\gamma, \dot{\gamma}, \xi) = \frac{1}{2}m(\xi_1^2 + \xi_2^2) + \frac{J_1 + J_2}{2}\xi_3^2 + J_2\xi_3\dot{\gamma} + \frac{J_2}{2}\dot{\gamma}^2.$$

Then the reduced Euler-Lagrange equations with controls (14) in this case are given by

$$\begin{aligned} m\dot{\xi}_1 &= u_1 \cos \gamma, \\ m\dot{\xi}_2 + (J_1 + J_2)\xi_1\xi_3 + J_2\xi_1\dot{\gamma} - m\xi_1\xi_3 &= u_1 \sin \gamma, \\ (J_1 + J_2)\dot{\xi}_3 + J_2\ddot{\gamma} - m\xi_2(\xi_1 + \xi_3) &= -u_1 p \sin \gamma, \\ J_2(\dot{\xi}_3 + \ddot{\gamma}) &= u_2. \end{aligned}$$

On the other hand, choosing the adapted basis $\{\mathcal{B}_a, \mathcal{B}_\alpha\}$ the modified equations of motion (16) read in this case as

$$\begin{aligned} m(\cos \gamma \dot{\xi}_1 + \sin \gamma (\dot{\xi}_2 - \xi_1\xi_3)) + (J_1 + J_2)\xi_1\xi_3 \sin \gamma + J_2\xi_1\dot{\gamma} \sin \gamma &= u_1, \\ m(\cos \gamma (\dot{\xi}_2 - \xi_1\xi_3) - \sin \gamma \dot{\xi}_1) + \xi_1\xi_3(J_1 + J_2) \cos \gamma + J_2\xi_1\dot{\gamma} \cos \gamma &= 0, \\ \frac{J_1 + J_2}{p}(\dot{\xi}_3 + p\xi_1\xi_3) + \frac{J_2}{p}(\ddot{\gamma} + p\xi_1\dot{\gamma}) + m(\dot{\xi}_2 - \xi_1\xi_3 - \frac{\xi_2\xi_1 + \xi_3\xi_2}{p}) &= 0, \\ J_2(\dot{\xi}_3 + \ddot{\gamma}) &= u_2. \end{aligned}$$

Now, we can study the optimal control problem that consists, as mentioned before, on finding a trajectory of state variables and control inputs satisfying the previous equations from given initial and final conditions $(\gamma(0), \dot{\gamma}(0), \xi(0))$, $(\gamma(T), \dot{\gamma}(T), \xi(T))$ respectively, and extremizing the cost functional

$$\int_0^T (\rho_1 u_1^2 + \rho_2 u_2^2) dt,$$

where ρ_1 and ρ_2 are constants.

The related optimal control problem is equivalent to a second-order Lagrangian problem with second-order constraints defined as follows (see [4] for more details). Extremize

$$\tilde{\mathcal{A}} = \int_0^T \tilde{L}(\xi, \dot{\xi}, \gamma, \dot{\gamma}, \ddot{\gamma}) dt,$$

subject to second-order constraints given by

$$\Phi^1 = m(\cos \gamma (\dot{\xi}_2 - \xi_1\xi_3) - \sin \gamma \dot{\xi}_1) + \xi_1\xi_3(J_1 + J_2) \cos \gamma + J_2\xi_1\dot{\gamma} \cos \gamma, \quad (18a)$$

$$\Phi^2 = \frac{J_1 + J_2}{p}(\dot{\xi}_3 + p\xi_1\xi_3) + \frac{J_2}{p}(\ddot{\gamma} + p\xi_1\dot{\gamma}) + m(\dot{\xi}_2 - \xi_1\xi_3 - \frac{\xi_2\xi_1 + \xi_3\xi_2}{p}). \quad (18b)$$

Here, $\tilde{L} : T^{(2)}S^1 \times 2\mathfrak{se}(2) \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned} \tilde{L}(\gamma, \dot{\gamma}, \ddot{\gamma}, \xi, \dot{\xi}) = & \\ \rho_1 \left(m(\cos \gamma \dot{\xi}_1 + \sin \gamma (\dot{\xi}_2 - \xi_1 \xi_3)) + (J_1 + J_2) \xi_1 \xi_3 \sin \gamma + J_2 \xi_1 \dot{\gamma} \sin \gamma \right)^2 & \\ + \rho_2 J_2^2 (\dot{\xi}_3 + \ddot{\gamma})^2, & \end{aligned} \quad (19)$$

which basically is the cost function $C = \rho_1 u_1^2 + \rho_2 u_2^2$ in terms of the new variables.

• **Discrete setting:** Following the prescription in theorem 4.1 and the further conclusion in corollary 4.2, we shall consider a discrete Lagrangian and discrete constraints when approaching the discrete associated problem. Moreover, since we are dealing with a constrained problem, we must include the constraints in the variational procedure as shown in §5.2. Therefore, the discrete Lagrangian and constraints read: $\tilde{L}_d : 3(S^1) \times 2SE(2) \rightarrow \mathbb{R}$, $\Phi_d^\alpha : 3(S^1) \times 2SE(2) \rightarrow \mathbb{R}$, $\alpha = 1, 2$. Furthermore, as was introduced in §2, the discrete Lagrangian \tilde{L}_d is chosen as a suitable approximation of the action sum $\int_0^T \tilde{L} dt$. Thus, we shall set consider

$$\begin{aligned} \tilde{L}_d(\gamma_k, \gamma_{k+1}, \gamma_{k+2}, W_k, W_{k+1}) + \lambda_\alpha^k \Phi_d^\alpha(\gamma_k, \gamma_{k+1}, \gamma_{k+2}, W_k, W_{k+1}) = & \\ = h \tilde{L} \left(\frac{\gamma_k + \gamma_{k+1} + \gamma_{k+2}}{3}, \frac{\gamma_{k+2} - \gamma_k}{2h}, \frac{\gamma_{k+2} - 2\gamma_{k+1} + \gamma_k}{h^2}, \frac{\tau^{-1}(W_k)}{h}, \frac{\tau^{-1}(W_{k+1}) - \tau^{-1}(W_k)}{h^2} \right) & \\ + \lambda_\alpha^k \Phi^\alpha \left(\frac{\gamma_k + \gamma_{k+1} + \gamma_{k+2}}{3}, \frac{\gamma_{k+2} - \gamma_k}{2h}, \frac{\gamma_{k+2} - 2\gamma_{k+1} + \gamma_k}{h^2}, \frac{\tau^{-1}(W_k)}{h}, \frac{\tau^{-1}(W_{k+1}) - \tau^{-1}(W_k)}{h^2} \right), & \end{aligned}$$

where τ is a general retraction map as shown in §2.1, \tilde{L} is defined in (19) and Φ^α are defined in (18). Here, $\gamma_k, \gamma_{k+1}, \gamma_{k+2} \in S^1$ while $W_k, W_{k+1} \in SE(2)$ (note that we are taking a symmetric approximation to γ_k , that is $\frac{\gamma_k + \gamma_{k+1} + \gamma_{k+2}}{3}$). In addition, we are taking the usual discretizations for the first and second derivatives, that is

$$\begin{aligned} \dot{\gamma}_k &\simeq \frac{\gamma_{k+2} - \gamma_k}{2h}, \\ \ddot{\gamma}_k &\simeq \frac{\gamma_{k+2} - 2\gamma_{k+1} + \gamma_k}{h^2}, \\ \dot{\xi}_k &\simeq \frac{\xi_{k+1} - \xi_k}{h}, \end{aligned}$$

where $\mathfrak{se}(2) \ni \xi_k = \tau^{-1}(W_k)/h$. Taking advantage of the retraction map, we can define the discrete Lagrangian and the discrete constraints on the Lie algebra, that is $\tilde{L}_d : 3(S^1) \times 2\mathfrak{se}(2) \rightarrow \mathbb{R}$, $\Phi_d^\alpha : 3(S^1) \times 2\mathfrak{se}(2) \rightarrow \mathbb{R}$, $\alpha = 1, 2$ (with some abuse of notation, we employ the same notation, that is \tilde{L}_d and Φ_d^α , for the Lagrangian and constraints in both spaces). To consider the Lie algebra instead of the Lie group shall be useful since the Lie algebra is a vector space and, moreover, we stay in the space where the original system, i.e. \tilde{L} , is defined. Namely

$$\begin{aligned} \tilde{L}_d(\gamma_k, \gamma_{k+1}, \gamma_{k+2}, \xi_k, \xi_{k+1}) + \lambda_\alpha^k \Phi_d^\alpha(\gamma_k, \gamma_{k+1}, \gamma_{k+2}, \xi_k, \xi_{k+1}) = & \\ = h \tilde{L} \left(\frac{\gamma_k + \gamma_{k+1} + \gamma_{k+2}}{3}, \frac{\gamma_{k+2} - \gamma_k}{2h}, \frac{\gamma_{k+2} - 2\gamma_{k+1} + \gamma_k}{h^2}, \frac{\xi_k + \xi_{k+1}}{2}, \frac{\xi_{k+1} - \xi_k}{h} \right) & \\ + \lambda_\alpha^k \Phi^\alpha \left(\frac{\gamma_k + \gamma_{k+1} + \gamma_{k+2}}{3}, \frac{\gamma_{k+2} - \gamma_k}{2h}, \frac{\gamma_{k+2} - 2\gamma_{k+1} + \gamma_k}{h^2}, \frac{\xi_k + \xi_{k+1}}{2}, \frac{\xi_{k+1} - \xi_k}{h} \right), & \end{aligned}$$

where again we take symmetric approximations to γ_k ($\frac{\gamma_k + \gamma_{k+1} + \gamma_{k+2}}{3}$) and ξ_k ($\frac{\xi_k + \xi_{k+1}}{2}$). Finally, as in §5.2, applying the usual calculus of variations we obtain the discrete equations of motion:

$$\begin{aligned} 0 = & D_1 \widetilde{L}_d|_\xi(\gamma_k, \gamma_{k+1}, \gamma_{k+2}) + \lambda_\alpha^k D_1 \Phi_d^\alpha|_\xi(\gamma_k, \gamma_{k+1}, \gamma_{k+2}) \\ & + D_2 \widetilde{L}_d|_\xi(\gamma_{k-1}, \gamma_k, \gamma_{k+1}) + \lambda_\alpha^{k-1} D_2 \Phi_d^\alpha|_\xi(\gamma_{k-1}, \gamma_k, \gamma_{k+1}) \\ & + D_3 \widetilde{L}_d|_\xi(\gamma_{k-2}, \gamma_{k-1}, \gamma_k) + \lambda_\alpha^{k-2} D_3 \Phi_d^\alpha|_\xi(\gamma_{k-2}, \gamma_{k-1}, \gamma_k), \end{aligned} \quad (20a)$$

$$\begin{aligned} 0 = & \text{Ad}_{\tau(h\xi_{k-1})}^*(d\tau_{h\xi_{k-1}}^{-1})^* \left(D_1 \widetilde{L}_d|_\gamma(\xi_{k-1}, \xi_k) + D_2 \widetilde{L}_d|_\gamma(\xi_{k-2}, \xi_{k-1}) \right) \\ & - (d\tau_{h\xi_k}^{-1})^* \left(D_1 \widetilde{L}_d|_\gamma(\xi_k, \xi_{k+1}) + D_2 \widetilde{L}_d|_\gamma(\xi_{k-1}, \xi_k) \right) \\ & + \text{Ad}_{\tau(h\xi_{k-1})}^*(d\tau_{h\xi_{k-1}}^{-1})^* \left(\lambda_\alpha^{k-1} D_1 \Phi_d^\alpha|_\gamma(\xi_{k-1}, \xi_k) + \lambda_\alpha^{k-2} D_2 \Phi_d^\alpha|_\gamma(\xi_{k-2}, \xi_{k-1}) \right) \\ & - (d\tau_{h\xi_k}^{-1})^* \left(\lambda_\alpha^k D_1 \Phi_d^\alpha|_\gamma(\xi_k, \xi_{k+1}) + \lambda_\alpha^{k-1} D_2 \Phi_d^\alpha|_\gamma(\xi_{k-1}, \xi_k) \right), \end{aligned} \quad (20b)$$

$$0 = \Phi_d^\alpha(\gamma_k, \gamma_{k+1}, \gamma_{k+2}, \xi_k, \xi_{k+1}), \quad k = 0, \dots, N-2; \quad \alpha = 1, 2. \quad (20c)$$

As before, the notation $\widetilde{L}_d|_\xi$, $\Phi_d^\alpha|_\xi$ denotes that the $\mathfrak{se}(2)$ are frozen while, correspondingly, $\widetilde{L}_d|_\gamma$, $\Phi_d^\alpha|_\gamma$ denotes that the S^1 variables are frozen. To derive (20b) the properties of the right-trivialized derivative of the retraction map (and inverse), defined in proposition 2.1, have been used (see [7, 26, 27]). In order to obtain the complete set of unknowns, that is $\gamma_{0:N}, \xi_{0:N}, \lambda_\alpha^{0:N-2}$, we also have to take into account the reconstruction equation, which in this case has the form

$$g_{k+1} = g_k \tau(h\xi_k), \quad (21)$$

where $g_k \in SE(2)$. Finally, the range of validity of equations (20a) and (20b) is $k = 2, \dots, N-2$.

As was established in §5.2, due to the variational procedure (γ_0, γ_1) and (γ_{N-1}, γ_N) are fixed, which leaves $\gamma_{2:N-2}$ as unknowns (i.e. $N-3$ unknowns). By the same variational procedure (g_0, g_1) and (g_{N-1}, g_N) are also fixed, which by means of (21) imply that ξ_0 and ξ_{N-1} are fixed. Nevertheless, due to the reconstruction discretization $g_{k+1} = g_k \tau(h\xi_k)$, is clear that fixing ξ_k implies constraints in the neighboring points, in this case g_{k+1} and g_k . If we allow ξ_N , that means constraints at the points g_{N+1} and g_N . Since we only consider time points up to $T = Nh$, having a constraint in the beyond-terminal configuration g_{N+1} makes no sense. Hence, to ensure that the effect of the terminal constraint on ξ is correctly accounted for, the set of algebra points must be reduced to $\xi_{0:N-1}$. Furthermore, since ξ_0 and ξ_{N-1} are also fixed, the final set of algebra unknowns reduces to $\xi_{1:N-2}$ (i.e. $3(N-2)$ unknowns, since $\dim \mathfrak{se}(2) = 3$).

On the other hand, the boundary condition $g(T)$ (recall that we are considering an optimal control problem), is enforced by the relation $\tau^{-1}(g_N^{-1}g(T)) = 0$, which basically means that $g_N = g(T)$. It is possible to translate this condition in terms of algebra elements as

$$\tau^{-1}(\tau(h\xi_{N-1})^{-1} \dots \tau(h\xi_0)^{-1} g_0^{-1} g(T)) = 0. \quad (22)$$

We have $2(N - 1)$ extra unknowns when adding the Lagrange multipliers $\lambda_\alpha^{0:N-2}$ (recall that, in this case $\alpha = 1, 2$). Summing up, we have

$$(N - 3) + 3(N - 2) + 2(N - 1)$$

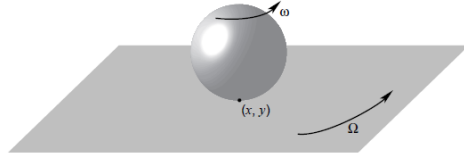
unknowns (corresponding to $\gamma_{2:N-2} + \xi_{1:N-2} + \lambda_\alpha^{0:N-2}$) for

$$(N - 3) + 3(N - 3) + 3 + 2(N - 1)$$

equations (corresponding to (20a)+(20b)+(22)+(20c)). Consequently, our discrete variational problem (which comes from the original optimal control problem) has become a nonlinear root finding problem. From the set $\xi_{0:N-1}$ we can reconstruct the configuration trajectory by means of the reconstruction equation (21). As was mentioned just after proposition 2.1, for computational reasons is useful to consider the retraction map τ as the Cayley map for $SE(2)$ instead of a truncation of the exponential map (see the Appendix for further details).

We also would like to stress that derivation of these discrete equations have a pure variational formulation and as a consequence, the integrators defined in this way are symplectic or Poisson-momentum preserving (see [37]). By using backward error analysis, it is well known that these integrators have a good energy behavior (see [42]). Similar techniques have been employed in [26] for the discrete optimal control of an underwater vehicle on $SE(3)$.

5.3.2. *Optimal Control of a Homogeneous Ball on a Rotating Plate.* We consider the following well-known problem (see [6, 11, 32, 28]), namely the model of a homogeneous ball on a rotating plate. A (homogeneous) ball of radius $r > 0$, mass m and inertia mk^2 about any axis rolls without sliding on a horizontal table which rotates with angular velocity Ω about a vertical axis x_3 through one of its points. Apart from the constant gravitational force, no other external forces are assumed to act on the sphere. Let (x, y) be denote the position of the point of contact of the sphere with the table. The configuration space of the sphere is $Q = \mathbb{R}^2 \times SO(3)$ where may be parametrized Q by (x, y, g) , $g \in SO(3)$, all measured with respect to the inertial frame. Let $\omega = (\omega_x, \omega_y, \omega_z)$ be the angular velocity vector of the sphere measured also with respect to the inertial frame. The potential energy is constant, so we may put $V = 0$.



The nonholonomic constraints are given by the non-sliding condition by

$$\begin{aligned}\dot{x} + \frac{r}{2} \text{Tr}(\dot{g}g^T E_2) &= -\Omega y, \\ \dot{y} - \frac{r}{2} \text{Tr}(\dot{g}g^T E_1) &= \Omega x,\end{aligned}$$

where $\{E_1, E_2, E_3\}$ is the standard basis of $\mathfrak{so}(3)$.

The matrix $\dot{g}g^T$ is skew-symmetric therefore we may write

$$\dot{g}g^T = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

where $(\omega_1, \omega_2, \omega_3)$ represents the angular velocity vector of the sphere measured with respect to the inertial frame. Then, we may rewrite the constraints in the usual form:

$$\begin{aligned}\dot{x} - r\omega_2 &= -\Omega y, \\ \dot{y} + r\omega_1 &= \Omega x.\end{aligned}$$

In addition, since we do not consider external forces the Lagrangian of the system corresponds with the kinetic energy

$$K(x, y, g, \dot{x}, \dot{y}, \dot{g}) = \frac{1}{2}(m\dot{x}^2 + m\dot{y}^2 + mk^2(\omega_1^2 + \omega_2^2 + \omega_3^2)).$$

Observe that the Lagrangian is metric on Q which is bi-invariant on $SO(3)$ as the ball is homogeneous.

Now, it is clear that $Q = \mathbb{R}^2 \times SO(3)$ is the total space of a trivial principal $SO(3)$ -bundle over \mathbb{R}^2 with respect the right $SO(3)$ -action given by $(x, y, R) \mapsto (x, y, RS)$ for all $S \in SO(3)$ and $(x, y, R) \in \mathbb{R}^2 \times SO(3)$. The action is in the right side since the symmetries are material symmetries.

The bundle projection $\phi : Q \rightarrow M = \mathbb{R}^2$ is just the canonical projection on the first factor. Therefore, we may consider the corresponding quotient bundle $E = TQ/SO(3)$ over $M = \mathbb{R}^2$. We will identify the tangent bundle to $SO(3)$ with $\mathfrak{so}(3) \times SO(3)$ by using right translation. Note that throughout the previous exposition we have employed the left trivialization. We use the right translation in this example for sake of simplicity. However, we would like to point out that the right trivialization just implies minor changes in the derivation of the equations of motion (see [24]).

Under this identification between $T(SO(3))$ and $\mathfrak{so}(3) \times SO(3)$, the tangent action of $SO(3)$ on $T(SO(3)) \cong \mathfrak{so}(3) \times SO(3)$ is the trivial action

$$(\mathfrak{so}(3) \times SO(3)) \times SO(3) \rightarrow \mathfrak{so}(3) \times SO(3), \quad ((\omega, g), h) \mapsto (\omega, gh). \quad (23)$$

Thus, the quotient bundle $TQ/SO(3)$ is isomorphic to the product manifold $T\mathbb{R}^2 \times \mathfrak{so}(3)$, and the vector bundle projection is $\tau_{\mathbb{R}^2} \circ pr_1$, where $pr_1 : T\mathbb{R}^2 \times \mathfrak{so}(3) \rightarrow T\mathbb{R}^2$ and $\tau_{\mathbb{R}^2} : T\mathbb{R}^2 \rightarrow \mathbb{R}^2$ are the canonical projections.

A section of $E = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathfrak{so}(3) \rightarrow \mathbb{R}^2$ is a pair (X, f) , where X is a vector field on \mathbb{R}^2 and $f : \mathbb{R}^2 \rightarrow \mathfrak{so}(3)$ is a smooth map. Therefore, a global basis

of sections of $T\mathbb{R}^2 \times \mathfrak{so}(3) \rightarrow \mathbb{R}^2$ is

$$\begin{aligned} e_1 &= \left(\frac{\partial}{\partial x}, 0\right), & e_2 &= \left(\frac{\partial}{\partial y}, 0\right), \\ e_3 &= (0, E_1), & e_4 &= (0, E_2), & e_5 &= (0, E_3). \end{aligned}$$

There exists a one-to-one correspondence between the space $\Gamma(E = TQ/SO(3))$ and the G -invariant vector fields on Q .

If $[\cdot, \cdot]$ is the Lie bracket on the space $\Gamma(E = TQ/SO(3))$, then the only non-zero fundamental Lie brackets are

$$[e_4, e_3] = e_5, \quad [e_5, e_4] = e_3, \quad [e_3, e_5] = e_4.$$

Moreover, it follows that the Lagrangian function $L = K$ and the constraints are $SO(3)$ -invariant. Consequently, L induces a Lagrangian function L' on $E = TQ/SO(3) \simeq T\mathbb{R}^2 \times \mathfrak{so}(3)$.

We have a constrained system on $E = TQ/SO(3) \simeq T\mathbb{R}^2 \times \mathfrak{so}(3)$ and note that in this case the constraints are nonholonomic and affine in the velocities. This kind of systems was recently analyzed by J. Cortés *et al* [14] (in particular, this example was carefully studied). The constraints define an affine subbundle of the vector bundle $E \simeq T\mathbb{R}^2 \times \mathfrak{so}(3) \rightarrow \mathbb{R}^2$ which is modeled over the vector subbundle \mathcal{D} generated by the sections

$$\mathcal{D} = \text{span}\{e_5; re_1 + e_4; re_2 - e_3\}$$

Moreover, the angular momentum of the ball about the axis x_3 is a conserved quantity since the Lagrangian is invariant under rotations about the axis x_3 and the infinitesimal generator for these rotations lies in the distribution \mathcal{D} . The conservation law is written as $\omega_z = c$, where c is a constant or as $\dot{\omega}_z = 0$. Then by the conservation of the angular momentum the second order constraints appear.

After some computations the equations of motion for this constrained system are precisely

$$\begin{cases} \dot{x} - r\omega_2 &= -\Omega y, \\ \dot{y} + r\omega_1 &= \Omega x, \\ \dot{\omega}_3 &= 0 \end{cases} \quad (24)$$

together with

$$\begin{aligned} \ddot{x} + \frac{k^2\Omega}{r^2 + k^2}\dot{y} &= 0, \\ \ddot{y} - \frac{k^2\Omega}{r^2 + k^2}\dot{x} &= 0. \end{aligned}$$

Now, we pass to an optimization problem. Assume full control over the motion of the center of the ball (the shape variables). The controlled system can be written as,

$$\begin{aligned} \ddot{x} + \frac{k^2\Omega}{r^2 + k^2}\dot{y} &= u_1, \\ \ddot{y} - \frac{k^2\Omega}{r^2 + k^2}\dot{x} &= u_2, \end{aligned}$$

subject to

$$\begin{cases} \omega_2 - \frac{1}{r}\dot{x} &= \frac{\Omega y}{r}, \\ \omega_1 + \frac{1}{r}\dot{y} &= \frac{\Omega x}{r}, \\ \dot{\omega}_3 &= 0. \end{cases} \quad (25)$$

Next, we consider the optimal control problem for this system following the techniques proposed in this paper.

Let C be the cost function in our optimization problem:

$$C = \frac{1}{2} (u_1^2 + u_2^2) .$$

we will want to minimize the total energy of the system. Given $q(0), q(T) \in \mathbb{R}^2$, $\dot{q}(0) \in T_{q(0)}\mathbb{R}^2$, $\dot{q}(T) \in T_{q(T)}\mathbb{R}^2$, $q = (x, y) \in \mathbb{R}^2$, $\omega(0), \omega(T) \in \mathfrak{so}(3)$, we look for an optimal control curve $(q(t), \omega(t), u(t))$ on the reduced space that steers the system from $(q(0), \omega(0))$ to $(q(T), \omega(T))$ minimizing

$$\int_0^T \frac{1}{2} (u_1^2 + u_2^2) dt,$$

subject to the constraints given by equations (25). (Recall also that $R(0), R(T) \in SO(3)$, the initial and final configurations of the problem, are also fixed. Its dynamics is given by the continuous reconstruction equation $\dot{R}(t) = R(t)\omega(t)$.)

As in the previous example, we define the second order Lagrangian $\tilde{L} : T^{(2)}\mathbb{R}^2 \times 2\mathfrak{so}(3) \rightarrow \mathbb{R}$ given by

$$\tilde{L}(x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}, \omega_1, \omega_2, \omega_3, \dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3) = \frac{1}{2} \left(\ddot{x} + \frac{k^2 \Omega}{r^2 + k^2} \dot{y} \right)^2 + \frac{1}{2} \left(\ddot{y} - \frac{k^2 \Omega}{r^2 + k^2} \dot{x} \right)^2 \quad (26)$$

subject to second-order constraints $\Phi^\alpha : T^{(2)}\mathbb{R}^2 \times 2\mathfrak{so}(3) \rightarrow \mathbb{R}$, $\alpha = 1, 2, 3$.

$$\Phi^1 = \omega_1 + \frac{1}{r}\dot{y} - \frac{\Omega x}{r}, \quad (27a)$$

$$\Phi^2 = \omega_2 - \frac{1}{r}\dot{x} - \frac{\Omega y}{r}, \quad (27b)$$

$$\Phi^3 = \dot{\omega}_3. \quad (27c)$$

As established in proposition 5.1, as a pure constrained variational problem, the optimal control problem is prescribed by solving the following system of 4-order

differential equations (ODEs).

$$\begin{aligned}
0 &= \lambda_1 \frac{\Omega}{r} + \frac{\dot{\lambda}_2}{r} + x^{(iv)} + \frac{2k^2\Omega\ddot{y}}{r^2 + k^2} - \frac{k^4\Omega^2\ddot{x}}{(r^2 + k^2)^2} \\
0 &= \lambda_2 \frac{\Omega}{r} + \frac{\dot{\lambda}_1}{r} + y^{(iv)} - \frac{2k^2\Omega\ddot{x}}{r^2 + k^2} - \frac{k^4\Omega^2\ddot{y}}{(r^2 + y^2)^2}, \\
0 &= \dot{\lambda}_1 + \lambda_2\omega_3 - \lambda_3\omega_2, \\
0 &= \dot{\lambda}_2 - \lambda_1\omega_3 + \lambda_3\omega_1, \\
0 &= \dot{\lambda}_3 + \lambda_1\omega_2 - \lambda_2\omega_1, \\
0 &= \omega_1 + \frac{1}{r}\dot{y} - \frac{\Omega x}{r}, \\
0 &= \omega_2 - \frac{1}{r}\dot{x} - \frac{\Omega y}{r}, \\
0 &= \dot{\omega}_3.
\end{aligned}$$

In addition, as mentioned before, the configurations $R \in SO(3)$ are given by the continuous reconstruction equation $\dot{R} = R\omega$.

In the particular case when the angular velocity Ω depends on the time (see [6, 34]), the equations of motion are rewritten as

$$\begin{aligned}
0 &= \lambda_1 \frac{\Omega(t)}{r} + \frac{\dot{\lambda}_2}{r} + x^{(iv)} + \frac{k^2\Omega''(t)\dot{y}}{r^2 + k^2} + \frac{2k^2\Omega'(t)\ddot{y}}{r^2 + k^2} + \frac{2k^2\Omega(t)\ddot{y}}{r^2 + k^2}, \\
&+ \frac{k^2\Omega'(t)\ddot{y}}{r^2 + k^2} - \frac{k^4\Omega^2(t)\ddot{x}}{(r^2 + k^2)^2} - \frac{2k^4\Omega'(t)\Omega(t)\dot{x}}{(r^2 + k^2)^2} \\
0 &= \lambda_2 \frac{\Omega(t)}{r} + \frac{\dot{\lambda}_1}{r} + y^{(iv)} - \frac{k^2\Omega''(t)\dot{x}}{r^2 + k^2} - \frac{3k^2\Omega'(t)\ddot{x}}{r^2 + k^2} - \frac{2k^2\Omega(t)\ddot{x}}{r^2 + k^2}, \\
&- \frac{k^4\Omega^2(t)\ddot{y}}{(r^2 + y^2)^2} - \frac{2k^4\Omega(t)\Omega'(t)\dot{y}}{(r^2 + k^2)^2}, \\
0 &= \dot{\lambda}_1 + \lambda_2\omega_3 - \lambda_3\omega_2, \\
0 &= \dot{\lambda}_2 - \lambda_1\omega_3 + \lambda_3\omega_1, \\
0 &= \dot{\lambda}_3 + \lambda_1\omega_2 - \lambda_2\omega_1, \\
0 &= \omega_1 + \frac{1}{r}\dot{y} - \frac{\Omega(t)x}{r}, \\
0 &= \omega_2 - \frac{1}{r}\dot{x} - \frac{\Omega(t)y}{r}, \\
0 &= \dot{\omega}_3.
\end{aligned}$$

• **Discrete setting:** As in the previous example, we discretize this problem by choosing a discrete Lagrangian \widetilde{L}_d and discrete constraints Φ_d^α . Employing equivalent arguments than in the previous example, we set $\widetilde{L}_d : 3(\mathbb{R}^2) \times 2\mathfrak{so}(3) \rightarrow \mathbb{R}$

and $\Phi_d^\alpha : 3(\mathbb{R}^2) \times 2\mathfrak{so}(3) \rightarrow \mathbb{R}$, $\alpha = 1, 2, 3$, as

$$\begin{aligned} & \widetilde{L}_d(q_k, q_{k+1}, q_{k+2}, \omega_k, \omega_{k+1}) + \lambda_\alpha^k \Phi_d^\alpha(q_k, q_{k+1}, q_{k+2}, \omega_k, \omega_{k+1}) = \\ & = h \widetilde{L}\left(\frac{q_k + q_{k+1} + q_{k+2}}{3}, \frac{q_{k+2} - q_k}{2h}, \frac{q_{k+2} - 2q_{k+1} + q_k}{h^2}, \frac{\omega_k + \omega_{k+1}}{2}, \frac{\omega_{k+1} - \omega_k}{h}\right) \\ & + \lambda_\alpha^k \Phi^\alpha\left(\frac{q_k + q_{k+1} + q_{k+2}}{3}, \frac{q_{k+2} - q_k}{2h}, \frac{q_{k+2} - 2q_{k+1} + q_k}{h^2}, \frac{\omega_k + \omega_{k+1}}{2}, \frac{\omega_{k+1} - \omega_k}{h}\right), \end{aligned}$$

We employ the same unknowns-equations counting process than in the previous example to find out that the number of unknowns matches the number of equations. Therefore, our discrete variational problem (which comes from the original optimal control problem) has become again a nonlinear root finding problem. For computational reasons is useful to consider the retraction map τ as the Cayley map for $SO(3)$ instead of a truncation of the exponential map (see the Appendix for further details).

6. CONCLUSIONS AND FUTURE WORK

6.1. Conclusions. In this paper, we have designed new variational integrators for the optimal control of underactuated mechanical systems, showing how developments in the theory of discrete mechanics and variational methods with constraints can be used to construct numerical optimal control algorithms with certain geometric desirable features. The methods are available for developing integrators on higher-order problems. The main idea is to use discrete variational calculus when the configuration space is a trivial principal bundle for systems with higher-order constraints and to derive the discrete Euler-Lagrange equation for discrete Lagrangians corresponding to a discretization of the higher-order Lagrangians.

It is also possible to use our techniques and the numeric integrators obtained for other interesting problems, for instance, the theory of k -splines on $SO(3)$ [22], [44]. In this paper, we show two applications of our ideas to the optimal control of mechanical systems defined on trivial principal bundles: an underactuated vehicle and a (homogeneous) ball rotating on a plate.

6.2. Future Work. A complete study of symmetry reduction, discrete Hamiltonian description, preservation of geometric structure and numerical simulations will be developed in a future paper. This discrete approach will be also studied and adapted to the Lie groupoid setting (see [15, 31, 37]). Another interesting point is to extend our methods to underactuated constrained systems using discrete constrained variational calculus (see [16] for the continuous counterpart). The case of optimal control problems for mechanical systems with nonholonomic constraints will be also studied using some of the ideas exposed through the paper (see [25] for more details).

APPENDIX: THE CAYLEY MAP

The Cayley map $\text{cay} : \mathfrak{g} \rightarrow G$ is defined by

$$\text{cay}(\xi) = \left(e - \frac{\xi}{2}\right)^{-1} \left(e + \frac{\xi}{2}\right)$$

and is valid for a general class of “quadratic groups” (see [23]) that include the groups of interest in this paper (e.g. $SO(3)$, $SE(2)$ and $SE(3)$). Its right trivialized derivative and inverse are defined by

$$\begin{aligned} \text{dcay}_x y &= (e - \frac{x}{2})^{-1} y (e + \frac{x}{2})^{-1}, \\ \text{dcay}_x^{-1} y &= (e - \frac{x}{2}) y (e + \frac{x}{2}). \end{aligned}$$

The exact form of the Cayley map for $SE(2)$ and $SO(3)$ is given in the following subsections.

The Cayley map for $SE(2)$. The coordinates of $SE(2)$ are (θ, x, y) with matrix representation $g \in SE(2)$ given by

$$g = \begin{pmatrix} \cos \theta & -\sin \theta & x \\ \sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the isomorphic map $\hat{\cdot} : \mathbb{R}^3 \rightarrow \mathfrak{se}(3)$ given by:

$$\hat{v} = \begin{pmatrix} 0 & -v_1 & v_2 \\ v_1 & 0 & v_3 \\ 0 & 0 & 0 \end{pmatrix},$$

where $v = (v_1, v_2, v_3)^T \in \mathbb{R}^3$. Thus, $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ can be used as a basis for $\mathfrak{se}(3)$, where $\{e_1, e_2, e_3\}$ is the standard basis of \mathbb{R}^3 . The map $\text{cay} : \mathfrak{se}(3) \rightarrow SE(2)$ is given by

$$\text{cay}(\hat{v}) = \begin{pmatrix} \frac{1}{4+v_1^2} \begin{pmatrix} 4-v_1^2 & -4v_1 & -2v_1v_3+4v_2 \\ 4v_1 & 4-v_1^2 & 2v_1v_2+4v_3 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix},$$

while the map $d\tau_\xi^{-1}$ becomes the 3×3 matrix

$$\text{dcay}_{\hat{v}}^{-1} = I_3 - \frac{1}{2}\text{ad}_v + \frac{1}{4}(v_1v \ 0_{3 \times 2}),$$

where

$$\text{ad}_v = \begin{pmatrix} 0 & 0 & 0 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}.$$

The Cayley map for $SO(3)$. the group of rigid body rotations is represented by 3×3 matrices with orthonormal column vectors corresponding to the axes of a right-handed frame attached to the body. On the other hand, the algebra $\mathfrak{so}(3)$ is the set of 3×3 antisymmetric matrices. A $\mathfrak{so}(3)$ basis can be constructed as $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$, $\hat{e}_i \in \mathfrak{so}(3)$, where $\{e_1, e_2, e_3\}$ is the standard basis for \mathbb{R}^3 . Elements $\xi \in \mathfrak{so}(3)$ can be identified with the vector $\omega \in \mathbb{R}^3$ through $\xi = \omega^\alpha \hat{e}_\alpha$, or $\xi = \hat{\omega}$. Under such identification the Lie bracket coincides with the standard cross product, i.e., $\text{ad}_{\hat{\omega}} \hat{\rho} = \omega \times \rho$, for some $\rho \in \mathbb{R}^3$. Using this identification and recalling the hat isomorphism $\hat{\cdot}$ defined above, we have

$$\text{cay}(\hat{\omega}) = I_3 + \frac{4}{4 + \|\omega\|^2} \left(\hat{\omega} + \frac{\hat{\omega}^2}{2} \right), \quad (28)$$

where I_3 is the 3×3 identity. The linear maps $d\tau_\xi$ and $d\tau_\xi^{-1}$ are expressed as the 3×3 matrices

$$\text{dcay}_\omega = \frac{2}{4 + \|\omega\|^2} (2I_3 + \hat{\omega}), \quad \text{dcay}_\omega^{-1} = I_3 - \frac{\hat{\omega}}{2} + \frac{\omega \omega^T}{4}. \quad (29)$$

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